

On absolute continuity of the spectrum of a d -dimensional periodic magnetic Dirac operator

L.I. Danilov

Physical-Technical Institute,
426000, Izhevsk, Kirov street, 132, Russia
e-mail: danilov@otf.pti.udm.ru

Abstract

In this paper, for $d \geq 3$, we prove the absolute continuity of the spectrum of a d -dimensional periodic Dirac operator with some discontinuous magnetic and electric potentials. In particular, for $d = 3$, electric potentials from Zygmund classes $L^3 \ln^{1+\delta} L(K)$, $\delta > 0$, and also ones with Coulomb singularities, with constraints on charges depending on the magnetic potential, are admitted (here K is the fundamental domain of the period lattice).

Introduction and main results

Let \mathcal{M}_M , $M \in \mathbb{N}$, be the linear space of complex $(M \times M)$ -matrices, let \mathcal{S}_M be the set of Hermitian matrices from \mathcal{M}_M , and let the matrices $\hat{\alpha}_j \in \mathcal{S}_M$, $j = 1, \dots, d$ ($d \geq 2$), satisfy the commutation relations $\hat{\alpha}_j \hat{\alpha}_l + \hat{\alpha}_l \hat{\alpha}_j = 2\delta_{jl} \hat{I}$, where $\hat{I} \in \mathcal{M}_M$ is the identity matrix and δ_{jl} is the Kronecker delta. Denote

$$\mathcal{S}_M^{(s)} = \{\hat{L} \in \mathcal{S}_M : \hat{L} \hat{\alpha}_j = (-1)^s \hat{\alpha}_j \hat{L} \text{ for all } j = 1, \dots, d\}, \quad s = 0, 1.$$

We consider the d -dimensional Dirac operator

$$\hat{\mathcal{D}} + \hat{W} = -i \sum_{j=1}^d \hat{\alpha}_j \frac{\partial}{\partial x_j} + \hat{W}(x), \quad x \in \mathbb{R}^d, \quad (0.1)$$

with a periodic matrix function $\hat{W} : \mathbb{R}^d \rightarrow \mathcal{S}_M$, $d \geq 2$ ($i^2 = -1$), with a period lattice $\Lambda \subset \mathbb{R}^d$. In particular, the operator (0.1) can have the form

$$\hat{\mathcal{D}} + \hat{W} = \sum_{j=1}^d \hat{\alpha}_j \left(-i \frac{\partial}{\partial x_j} - A_j \right) + \hat{V}, \quad \hat{V} = \hat{V}^{(0)} + \hat{V}^{(1)}, \quad (0.2)$$

where the components A_j of the magnetic potential $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the matrix functions $\hat{V}^{(s)} : \mathbb{R}^d \rightarrow \mathcal{S}_M^{(s)}$, $s = 0, 1$, are also periodic with the period lattice $\Lambda \subset \mathbb{R}^d$. In the sequel,

the matrix functions $\widehat{V}^{(s)}$, $s = 0, 1$, will be also chosen in the form

$$\widehat{V}^{(0)} = V\widehat{I}, \quad \widehat{V}^{(1)} = V_1\widehat{\beta}, \quad (0.3)$$

where V, V_1 are Λ -periodic real-valued functions and $\widehat{\beta} \in \mathcal{S}_M^{(1)}$ is a Hermitian matrix with $\widehat{\beta}^2 = \widehat{I}$, and in the particular form

$$\widehat{V}^{(0)} = V\widehat{I}, \quad \widehat{V}^{(1)} = m\widehat{\beta}, \quad (0.4)$$

where $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Λ -periodic electric potential and $m \in \mathbb{R}$.

The coordinates in \mathbb{R}^d are taken relative to an orthogonal basis $\{\mathcal{E}_j\}$ ($|\mathcal{E}_j| = 1$, $j = 1, \dots, d$; $|\cdot|$ and (\cdot, \cdot) are the length and the scalar product of vectors in \mathbb{R}^d), $A_j(x) = (A(x), \mathcal{E}_j)$, $x \in \mathbb{R}^d$. Let $\{E_j\}$ be the basis in the lattice $\Lambda \subset \mathbb{R}^d$,

$$K = \{x = \sum_{j=1}^d \xi_j E_j : 0 \leq \xi_j < 1, \quad j = 1, \dots, d\}.$$

Denote by $v(\cdot)$ the Lebesgue measure on \mathbb{R}^d ; $v(K)$ is the volume of the fundamental domain K . In what follows, the functions defined on the fundamental domain K will be also identified with their Λ -periodic extensions to \mathbb{R}^d .

The scalar products and the norms on the spaces \mathbb{C}^M , $L^2(\mathbb{R}^d; \mathbb{C}^M)$, and $L^2(K; \mathbb{C}^M)$ are introduced in the usual way (as a rule, omitting the notation for the corresponding space). We assume that the scalar products are linear in the second argument. For matrices $\widehat{L} \in \mathcal{M}_M$, we write

$$\|\widehat{L}\|_{\mathcal{M}_M} = \max_{u \in \mathbb{C}^M : \|u\|=1} \|\widehat{L}u\|.$$

The zero and the identity matrices and operators in various spaces are denoted by $\widehat{0}$ and \widehat{I} , respectively.

Let $H^1(\mathbb{R}^d; \mathbb{C}^M)$ be the Sobolev class (of order 1) of vector functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}^M$. The operator

$$\widehat{\mathcal{D}} = -i \sum_{j=1}^d \widehat{\alpha}_j \frac{\partial}{\partial x_j}$$

acts on the space $L^2(\mathbb{R}^d; \mathbb{C}^M)$ and has the domain $D(\widehat{\mathcal{D}}) = H^1(\mathbb{R}^d; \mathbb{C}^M)$. For $a \geq 0$, let $\mathbb{L}_M^\Lambda(d; a)$ be the set of Λ -periodic matrix functions $\widehat{W} \in L_{\text{loc}}^2(\mathbb{R}^d; \mathcal{M}_M)$ which have bounds $b(\widehat{W}) \leq a$ relative to the operator $\widehat{\mathcal{D}}$. If $\widehat{W} \in \mathbb{L}_M^\Lambda(d; a)$ and $\varphi \in H^1(\mathbb{R}^d; \mathbb{C}^M)$, then $\widehat{W}\varphi \in L^2(\mathbb{R}^d; \mathbb{C}^M)$ and for any $\varepsilon > 0$ there is a number $C_\varepsilon(a, \widehat{W}) > 0$ such that for all vector functions $\varphi \in H^1(\mathbb{R}^d; \mathbb{C}^M)$ the estimate

$$\|\widehat{W}\varphi\| \leq (a + \varepsilon) \|\widehat{\mathcal{D}}\varphi\| + C_\varepsilon(a, \widehat{W}) \|\varphi\|. \quad (0.5)$$

holds. In particular, the set $\mathbb{L}_M^\Lambda(d; 0)$ (with $a = 0$) contains Λ -periodic matrix functions $\widehat{W} : \mathbb{R}^d \rightarrow \mathcal{M}_M$ for which at least one of the following conditions is satisfied:

- 1) $d = 2$, the function $\|\widehat{W}(\cdot)\|_{\mathcal{M}_M}^2$ belongs to the Kato class K_2 (see [1]); this condition is fulfilled for the functions \widehat{W} from the Zygmund class $L^2 \ln L(K; \mathcal{M}_M)$;
 2) $d \geq 2$, $\widehat{W} \in L^2(K; \mathcal{M}_M)$ and

$$||| \widehat{W} |||_{\gamma, M} \doteq \operatorname{ess\,sup}_{x \in \mathbb{R}^d} \left(\int_0^1 \|\widehat{W}(x - \xi\gamma)\|_{\mathcal{M}_M}^2 d\xi \right)^{\frac{1}{2}} < +\infty$$

for some vector $\gamma \in \Lambda \setminus \{0\}$ (see, e.g., [2]);

- 3) $d \geq 3$, $\widehat{W} \in L^d(K; \mathcal{M}_M)$.

Let $L_w^d(K; \mathcal{M}_M)$ be the space of functions $\widehat{W} : K \rightarrow \mathcal{M}_M$ for which

$$\|\widehat{W}\|_{L_w^d(K; \mathcal{M}_M)} \doteq \sup_{t > 0} t (v(\{x \in K : \|\widehat{W}(x)\|_{\mathcal{M}_M} > t\}))^{\frac{1}{d}} < +\infty.$$

For functions $\widehat{W} \in L_w^d(K; \mathcal{M}_M)$, we write

$$\|\widehat{W}\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty)} \doteq \limsup_{t \rightarrow +\infty} t (v(\{x \in K : \|\widehat{W}(x)\|_{\mathcal{M}_M} > t\}))^{\frac{1}{d}}.$$

For $d \geq 3$, the Λ -periodic function $\widehat{W} \in L_w^d(K; \mathcal{M}_M)$ has the bound

$$b(\widehat{W}) \leq C \|\widehat{W}\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty)}$$

relative to the operator $\widehat{\mathcal{D}}$, where $C = C(d) > 0$ (see, e.g., [3]). From this one also derives the estimate

$$b(\widehat{W}) \leq C \|\widehat{W}\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty, \text{loc})}, \quad (0.6)$$

where

$$\|\widehat{W}\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \doteq \lim_{r \rightarrow +0} \sup_{x \in \mathbb{R}^d} \limsup_{t \rightarrow +\infty} t (v(\{y \in B_r(x) : \|\widehat{W}(y)\|_{\mathcal{M}_M} > t\}))^{\frac{1}{d}},$$

$B_r(x) = \{y \in \mathbb{R}^d : |x - y| \leq r\}$. If $\widehat{W}_1, \widehat{W}_2 \in L_w^d(K; \mathcal{M}_M)$, then

$$\|\widehat{W}_1 + \widehat{W}_2\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \leq 2 \|\widehat{W}_1\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty, \text{loc})} + 2 \|\widehat{W}_2\|_{L_w^d(K; \mathcal{M}_M)}^{(\infty, \text{loc})}.$$

Let $\{E_j^*\}$ be the basis in the reciprocal lattice $\Lambda^* \subset \mathbb{R}^d$, $(E_j, E_l^*) = \delta_{jl}$. We let

$$\psi_N = v^{-1}(K) \int_K \psi(x) e^{-2\pi i(N, x)} dx, \quad N \in \Lambda^*,$$

denote the Fourier coefficients of the functions $\psi \in L^1(K; \mathcal{U})$, where \mathcal{U} is the space \mathbb{C}^M or \mathbb{R}^d or \mathcal{M}_M .

If $\widehat{W} : \mathbb{R}^d \rightarrow \mathcal{S}_M$ is a Hermitian matrix function and $\widehat{W} \in \mathbb{L}_M^\Lambda(d; a)$ for some $a \in [0, 1)$, then $\widehat{\mathcal{D}} + \widehat{W}$ is a self-adjoint operator on $L^2(\mathbb{R}^d; \mathbb{C}^M)$ with the domain $D(\widehat{\mathcal{D}} + \widehat{W}) = D(\widehat{\mathcal{D}}) = H^1(\mathbb{R}^d; \mathbb{C}^M)$ (see [3, 4]). The singular spectrum of the operator $\widehat{\mathcal{D}} + \widehat{W}$ is empty and the eigenvalues (if they exist) have an infinite multiplicity and form a discrete set (see

[5] and also [6]). Therefore, if there are no eigenvalues in the spectrum of the operator $\widehat{D} + \widehat{W}$, then the spectrum is absolutely continuous (this assertion is also a consequence of the results of [7]).

The question on the absolute continuity of the spectrum of periodic operators of mathematical physics (in particular, of the periodic Dirac operator) attracted a lot of attention in the past decade. Two papers [8] and [9] contain a survey of some early results. The assertions on the absolute continuity of the spectrum of periodic Schrödinger operators (including ones with variable metrics) can be found in [10] – [25] (also see references therein). The periodic Maxwell operator was considered in [26, 27].

The first results on the absolute continuity of the spectrum of the periodic Dirac operator were obtained in [28, 29, 30]. In [30, 31], the absolute continuity of the spectrum of the operator (0.2), (0.4) was proved for all $d \geq 2$ under the conditions $V \in C(\mathbb{R}^d)$, $A \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$, and

$$\| |A| \|_{L^\infty(\mathbb{R}^d)} < \max_{\gamma \in \Lambda \setminus \{0\}} \frac{\pi}{|\gamma|}. \quad (0.7)$$

In subsequent papers, the restriction on the periodic electric potential V has been relaxed. The spectrum of the operator (0.2), (0.4) is absolutely continuous if at least one of the following conditions is satisfied:

1) $d = 2$, $V \in L^q(K)$, $q > 2$, and the magnetic potential $A \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ obeys condition (0.7) (see [32]);

2) $d \geq 3$, $A \equiv 0$, and $\sum_{N \in \Lambda^*} |V_N|^p < +\infty$, where $p \in [1, q_d(q_d - 1)^{-1})$ and the numbers $q_d > d$ are found as the largest roots of the algebraic equations

$$q^4 - (3d^2 - 4d - 1)q^3 + 2(4d^2 - 6d - 3)q^2 - (9d^2 - 16d - 4)q - 4d(d - 2) = 0,$$

$q_3 \simeq 11.645$, $d^{-2}q_d \rightarrow 3$ as $d \rightarrow +\infty$ (see [5] and also [29, 31, 32]);

3) $d = 3$, $V \in L^q(K)$, $q > 3$, and the magnetic potential $A \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ satisfies (0.7) (see [33]);

4) $d \geq 2$, $V \in L^2(K)$, $A \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$, and there exists a vector $\gamma \in \Lambda \setminus \{0\}$ such that $\| |A| \|_{L^\infty(\mathbb{R}^d)} < \pi |\gamma|^{-1}$ and the map

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow V(x - \xi\gamma)\} \in L^2([0, 1])$$

is continuous (see [2]).

In [34], the absolute continuity of the spectrum of the operator (0.2), (0.4) was proved for $d = 3$ under conditions: the matrix functions $\widehat{V}^{(s)}$, $s = 0, 1$, belong to the Zygmund class $L^3 \ln^{2+\delta} L(K; \mathcal{M}_M)$ for some $\delta > 0$, and the magnetic potential $A \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ satisfies (0.7).

In recent paper [35], it was proved that the spectrum of the Dirac operator (0.1) is absolutely continuous if

$$\widehat{\alpha}_1 \widehat{W} \widehat{\alpha}_1 = \widehat{\alpha}_2 \widehat{W} \widehat{\alpha}_2 = \dots = \widehat{\alpha}_d \widehat{W} \widehat{\alpha}_d$$

and for some $r \geq d$, $\alpha > (d - 1)/(2r)$, we have $\widehat{W} \in L^r(K; \mathcal{S}_M)$ and

$$\| \widehat{W}(\cdot + y) - \widehat{W}(\cdot) \|_{L^r(K; \mathcal{S}_M)} \leq \widetilde{C} \{\text{dist}(y, \Lambda)\}^\alpha$$

for any $y \in \mathbb{R}^d$, where $\tilde{C} \geq 0$ and

$$\text{dist}(y, \Lambda) = \min_{\gamma \in \Lambda} |y - \gamma|.$$

For $d = 3$, one can set

$$\widehat{\beta} = \begin{pmatrix} \widehat{I} & \widehat{0} \\ \widehat{0} & -\widehat{I} \end{pmatrix}, \quad \widehat{\alpha}_j = \begin{pmatrix} \widehat{0} & \widehat{\sigma}_j \\ \widehat{\sigma}_j & \widehat{0} \end{pmatrix}, \quad j = 1, 2, 3, \quad (0.8)$$

where $\widehat{0}$ and \widehat{I} are the zero and the identity 2×2 matrices, and $\widehat{\sigma}_j$ are the Pauli matrices:

$$\widehat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \widehat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \widehat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, the matrix functions $\widehat{V}^{(s)} : \mathbb{R}^3 \rightarrow \mathcal{S}_4^{(s)}$, $s = 0, 1$, can be chosen in the form

$$\widehat{V}^{(0)} = V_1^{(0)} \widehat{I} - i V_2^{(0)} \widehat{\alpha}_1 \widehat{\alpha}_2 \widehat{\alpha}_3, \quad \widehat{V}^{(1)} = V_1^{(1)} \widehat{\beta} + V_2^{(1)} \widehat{\alpha}_1 \widehat{\alpha}_2 \widehat{\alpha}_3 \widehat{\beta},$$

where

$$-i \widehat{\alpha}_1 \widehat{\alpha}_2 \widehat{\alpha}_3 = \begin{pmatrix} \widehat{0} & \widehat{I} \\ \widehat{I} & \widehat{0} \end{pmatrix}, \quad \widehat{\alpha}_1 \widehat{\alpha}_2 \widehat{\alpha}_3 \widehat{\beta} = \begin{pmatrix} \widehat{0} & -i \widehat{I} \\ i \widehat{I} & \widehat{0} \end{pmatrix},$$

and $V_l^{(s)}$, $l = 1, 2$, are Λ -periodic real-valued functions.

For $d = 2$, one can identify the matrices $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, and $\widehat{\beta}$ with the Pauli matrices $\widehat{\sigma}_1$, $\widehat{\sigma}_2$, and $\widehat{\sigma}_3$, respectively.

The two-dimensional periodic Dirac operator (0.2), (0.3) with an unbounded magnetic potential A was studied in [36, 37]. In [37], the absolute continuity of the spectrum of the operator (0.2), (0.3) (with $d = 2$) was proved under the conditions $V, V_1 \in L^q(K)$ and $A \in L^q(K; \mathbb{R}^2)$, $q > 2$. A similar result was obtained in [36] (it was assumed, however, that $V_1 \equiv m = \text{const}$, but the proof carries over to functions $V_1 \in L^q(K)$, $q > 2$, without essential modifications). The methods used in [36] were the same as in [32]. More general conditions on V , V_1 , and A were obtained in [38]: it suffices to require that the functions $V^2 \ln(1 + |V|)$, $V_1^2 \ln(1 + |V_1|)$, and $|A|^2 \ln^{1+\delta}(1 + |A|)$ belong to the space $L^1(K)$ for some $\delta > 0$.

In [24, 39], it was proved that there are no eigenvalues in the spectrum of a generalized two-dimensional periodic Dirac operator

$$-i \sum_{j=1}^2 (h_{j1} \widehat{\sigma}_1 + h_{j2} \widehat{\sigma}_2) \frac{\partial}{\partial x_j} + \widehat{W}, \quad (0.9)$$

where $h_{jl} \in L^\infty(\mathbb{R}^2; \mathbb{R})$, $j, l = 1, 2$, are Λ -periodic functions, for which

$$0 < \varepsilon \leq h_{11}(x)h_{22}(x) - h_{12}(x)h_{21}(x)$$

for a.e. $x \in \mathbb{R}^2$, and $\widehat{W} \in \mathbb{L}_2^\Lambda(2; 0)$. If the operator (0.9) is self-adjoint, then its spectrum is absolutely continuous. Some particular cases of the operator (0.9) were also considered

in [14, 40, 41] (in [40], the functions h_{jl} were supposed to obey the same conditions as in [39], but it was assumed that $\widehat{W} \in L^q(K; \mathcal{M}_M)$, $q > 2$).

In [37], the absolute continuity of the spectrum of the d -dimensional operator (0.2), (0.3) was proved for $d \geq 3$ under the conditions $V, V_1 \in C(\mathbb{R}^d; \mathbb{R})$ and $A \in C^{2d+3}(\mathbb{R}^d; \mathbb{R}^d)$. The proof was based on Sobolev's paper [13], where the absolute continuity of the spectrum was proved for the Schrödinger operator with a periodic magnetic potential $A \in C^{2d+3}(\mathbb{R}^d; \mathbb{R}^d)$, $d \geq 3$. The last condition was relaxed by Kuchment and Levendorskii in [9]: it suffices to require that $A \in H_{\text{loc}}^q(\mathbb{R}^d; \mathbb{R}^d)$, $2q > 3d - 2$, which makes it possible to relax accordingly the constraint on the magnetic potential A also for the periodic Dirac operator (see [8, 37]).

Let $S_{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. For vectors $x \in \mathbb{R}^d \setminus \{0\}$, we shall use the notation

$$S_{d-2}(x) = \{\tilde{e} \in S_{d-1} : (\tilde{e}, x) = 0\}.$$

Let $\mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, be the set of all even Borel signed measures μ on \mathbb{R} (with finite total variation) for which $\int_{\mathbb{R}} e^{ipt} d\mu(t) = 1$ for every $p \in (-\mathfrak{h}, \mathfrak{h})$;

$$\|\mu\| = \sup_{\mathcal{O} \in \mathcal{B}(\mathbb{R})} (|\mu(\mathcal{O})| + |\mu(\mathbb{R} \setminus \mathcal{O})|) < +\infty, \quad \mu \in \mathfrak{M}_{\mathfrak{h}},$$

where $\mathcal{B}(\mathbb{R})$ is the collection of Borel subsets $\mathcal{O} \subseteq \mathbb{R}$. In [42, 43], it was proved that the spectrum of the Λ -periodic Dirac operator (0.2) is absolutely continuous for $d \geq 3$ if the following conditions are fulfilled:

- 1) $\widehat{V}^{(s)} \in C(\mathbb{R}^d; \mathcal{S}_M^{(s)})$, $s = 0, 1$;
- 2) $A \in C(\mathbb{R}^d; \mathbb{R}^d)$ and there exist a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, such that for every $x \in \mathbb{R}^d$ and every unit vector $\tilde{e} \in S_{d-2}(\gamma)$ we have

$$\left| A_0 - \int_{\mathbb{R}} d\mu(t) \int_0^1 A(x - \xi\gamma - t\tilde{e}) d\xi \right| < \frac{\pi}{|\gamma|}, \quad (0.10)$$

where $A_0 = v^{-1}(K) \int_K A(x) dx$.

For the periodic magnetic potential $A \in C(\mathbb{R}^d; \mathbb{R}^d)$, $d \geq 3$, condition (0.10) is fulfilled (under an appropriate choice of $\gamma \in \Lambda \setminus \{0\}$ and $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$) whenever $A \in H_{\text{loc}}^q(\mathbb{R}^d; \mathbb{R}^d)$, $2q > d - 2$, and also in the case where $\sum_{N \in \Lambda^*} \|A_N\|_{\mathbb{C}^d} < +\infty$ (see [42, 43]).

Let a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, be fixed. Denote $e \doteq |\gamma|^{-1}\gamma \in S_{d-1}$. In this paper, we consider the Dirac operator (0.2) for $d \geq 3$ supposing that the Λ -periodic magnetic potential $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies the following conditions (\mathbf{A}_0) , (\mathbf{A}_1) , $(\widehat{\mathbf{A}}_1)$, and (\mathbf{A}_2) .

(\mathbf{A}_0) : $A_0 = 0$.

Since one always can make the transformation

$$\widehat{\mathcal{D}} + \widehat{W} \rightarrow e^{i(A_0, x)} (\widehat{\mathcal{D}} + \widehat{W}) e^{-i(A_0, x)},$$

without loss of generality we can assume that condition (\mathbf{A}_0) holds.

(**A**₁): $A \in L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ and the map

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow A(x - \xi\gamma)\} \in L^2([0, 1]; \mathbb{R}^d)$$

is continuous (in particular, this means that for all $x \in \mathbb{R}^d$, the function $\xi \rightarrow A(x - \xi\gamma)$ is defined for a.e. $\xi \in \mathbb{R}$).

Since for any $\varepsilon > 0$ there exists a number $C'(\gamma, \varepsilon) > 0$ such that for all Λ -periodic matrix functions $\widehat{W} \in L^2_{\text{loc}}(\mathbb{R}^d; \mathcal{M}_M)$ (for which $|||\widehat{W}|||_{\gamma, M} < +\infty$) and for all vector functions $\varphi \in H^1(\mathbb{R}^d; \mathbb{C}^M)$ the inequality

$$|||\widehat{W}\varphi||| \leq |||\widehat{W}|||_{\gamma, M} (\varepsilon \left\| -i \frac{\partial \varphi}{\partial x'_2} \right\| + C'(\gamma, \varepsilon) \|\varphi\|) \quad (0.11)$$

holds, where $x'_2 \doteq (x, e)$, $x \in \mathbb{R}^d$ (we can put $C'(\gamma, \varepsilon) = 2(1 + 2\varepsilon^{-1}|\gamma|^2)$; see, e.g., [2]), condition (**A**₁) implies that

$$\sum_{j=1}^d A_j \widehat{\alpha}_j \in \mathbb{L}_M^\Lambda(d; 0).$$

The following condition is a consequence of condition (**A**₁):

(**A**₁[~]): there is a constant $C^* > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \sup_{\tilde{e} \in S_{d-2}(\gamma)} \iint_{\xi_1^2 + \xi_2^2 \leq 1} |A(x - \xi_1 \tilde{e} - \xi_2 e)| \frac{d\xi_1 d\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} \leq C^*. \quad (0.12)$$

Indeed, for all $x \in \mathbb{R}^d$ and all $\tilde{e} \in S_{d-2}(\gamma)$,

$$\begin{aligned} & \iint_{\xi_1^2 + \xi_2^2 \leq 1} |A(x - \xi_1 \tilde{e} - \xi_2 e)| \frac{d\xi_1 d\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} \leq \\ & \leq \int_{|\xi_1| \leq 1} \left(\int_{-1}^1 |A(x - \xi_1 \tilde{e} - \xi_2 e)|^2 d\xi_2 \right)^{\frac{1}{2}} \sqrt{\frac{\pi}{|\xi_1|}} d\xi_1 \leq \\ & \leq 4\sqrt{\pi} \left(-\left[-\frac{2}{|\gamma|} \right] \cdot |\gamma| \right)^{\frac{1}{2}} \left(\max_{y \in \mathbb{R}^d} \int_0^1 |A(y - \xi\gamma)|^2 d\xi \right)^{\frac{1}{2}}, \end{aligned}$$

where $[t]$ is the integral part of a number $t \in \mathbb{R}$. Therefore, we can put

$$C^* = 4\sqrt{\pi} \left(-\left[-\frac{2}{|\gamma|} \right] \cdot |\gamma| \right)^{\frac{1}{2}} \left(\max_{y \in \mathbb{R}^d} \int_0^1 |A(y - \xi\gamma)|^2 d\xi \right)^{\frac{1}{2}}.$$

Let us denote

$$\widetilde{A}(\tilde{e}; x) = \widetilde{A}(\gamma, \mu, \tilde{e}; x) = \int_{\mathbb{R}} d\mu(t) \int_0^1 A(x - \xi\gamma - t\tilde{e}) d\xi, \quad x \in \mathbb{R}^d, \quad \tilde{e} \in S_{d-2}(\gamma).$$

From condition (\mathbf{A}_1) it follows that the periodic function

$$\mathbb{R}^d \ni x \rightarrow \int_0^1 A(x - \xi\gamma) d\xi$$

is continuous. Therefore the function $\tilde{A}(\cdot; \cdot) : S_{d-2}(\gamma) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is also continuous, and Λ -periodic in the second argument. Moreover, $(\tilde{A}(\tilde{e}; \cdot))_0 = A_0 = 0$ (for all $\tilde{e} \in S_{d-2}(\gamma)$).

(\mathbf{A}_2) : *there is a constant $\tilde{\theta} \in [0, 1)$ such that*

$$\max_{x \in \mathbb{R}^d} \max_{\tilde{e} \in S_{d-2}(\gamma)} |\tilde{A}(\tilde{e}; x)| \leq \frac{\tilde{\theta}\pi}{|\gamma|}. \quad (0.13)$$

If we pick the Dirac measure $\mu = \delta$, then the function $\tilde{A}(\tilde{e}; \cdot)$ does not depend on the vector \tilde{e} and inequality (0.13) means that

$$\max_{x \in \mathbb{R}^d} \left| \int_0^1 A(x - \xi\gamma) d\xi \right| \leq \frac{\tilde{\theta}\pi}{|\gamma|}.$$

The last inequality is valid if

$$\sum_{N \in \Lambda^* : (N, \gamma) = 0} \|A_N\|_{\mathbb{C}^d} \leq \frac{\tilde{\theta}\pi}{|\gamma|}.$$

The following two theorems are the main results of this paper.

Theorem 0.1. *Suppose $d \geq 3$, $\widehat{V}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, $A \in L^2(K; \mathbb{R}^d)$, and there exist a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, such that conditions (\mathbf{A}_0) , (\mathbf{A}_1) , (\mathbf{A}_2) are fulfilled for the magnetic potential A , and the maps*

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow \widehat{V}^{(s)}(x - \xi\gamma)\} \in L^2([0, 1]; \mathcal{S}_M^{(s)}), \quad s = 0, 1,$$

are continuous. Then the spectrum of the periodic Dirac operator (0.2) is absolutely continuous.

Denote

$$x_{\parallel} = (x, e)e, \quad x_{\perp} = x - (x, e)e, \quad x \in \mathbb{R}^d.$$

For a matrix function $\widehat{W} \in L^2(K; \mathcal{M}_M)$ and a number $\sigma \in [0, 2]$, we set

$$\beta_{\gamma, \sigma}(R; \widehat{W}) \doteq v(K) \sup_{N \in \Lambda^* : 2\pi|N_{\perp}| \geq R} (2\pi|N_{\perp}|)^{2-\sigma} (2\pi|N|)^{\sigma} \|\widehat{W}_N\|, \quad R \geq 0,$$

$$\beta_{\gamma, \sigma}(\widehat{W}) \doteq \lim_{R \rightarrow +\infty} \beta_{\gamma, \sigma}(R; \widehat{W}).$$

Let $\text{supp } \widehat{W}$ be the essential support of a measurable function $\widehat{W} : \mathbb{R}^d \rightarrow \mathcal{M}_M$, $\text{supp } \widehat{W} = \mathbb{R}^d \setminus \{x \in \mathbb{R}^d : \widehat{W}(y) = \widehat{0} \text{ for a.e. } y \in B_r(x) \text{ for some } r = r(x) > 0\}$.

Theorem 0.2. Suppose $d = 3$, $\widehat{V}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, $A \in L^2(K; \mathbb{R}^3)$, and there exist a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, such that conditions (\mathbf{A}_0) , (\mathbf{A}_1) , (\mathbf{A}_2) are satisfied for the magnetic potential A , and the functions $\widehat{V}^{(s)}$, $s = 0, 1$, can be represented in the form

$$\widehat{V}^{(s)} = \widehat{V}_1^{(s)} + \widehat{V}_2^{(s)} + \widehat{V}_3^{(s)},$$

where (for $s = 0, 1$) the Λ -periodic matrix functions $\widehat{V}_\nu^{(s)}$, $\nu = 1, 2, 3$, obey the conditions:

1) for the functions $\widehat{V}_1^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, the maps

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow \widehat{V}_1^{(s)}(x - \xi\gamma)\} \in L^2([0, 1]; \mathcal{S}_M^{(s)})$$

are continuous;

2) $\widehat{V}_2^{(s)} \in L^3 \ln^{1+\delta} L(K; \mathcal{S}_M^{(s)})$ for some $\delta > 0$;

3) $\widehat{V}_3^{(s)} = \sum_{q=1}^{Q_s} \widehat{V}_{3,q}^{(s)}$, $\widehat{V}_{3,q}^{(s)} \in L_w^3(K; \mathcal{S}_M^{(s)})$, and $\beta_{\gamma, \sigma}(0; \widehat{V}_{3,q}^{(s)}) < +\infty$ for some $\sigma \in (0, 2]$, $q = 1, \dots, Q_s$, moreover, the essential supports $\text{supp } \widehat{V}_{3,q}^{(s)}$ of the functions $\widehat{V}_{3,q}^{(s)}$ (considered as Λ -periodic functions defined on \mathbb{R}^3) do not intersect for different q ,

$$\|\widehat{V}_3^{(s)}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} = \max_{q=1, \dots, Q_s} \|\widehat{V}_{3,q}^{(s)}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \leq c_1 \quad \text{and} \quad \max_{q=1, \dots, Q_s} \beta_{\gamma, \sigma}(\widehat{V}_{3,q}^{(s)}) \leq c'_1,$$

where $c_1 = c_1(\mathfrak{h}, \mu, \gamma; A) \in (0, C^{-1})$ and $c'_1 = c'_1(\mathfrak{h}, \mu, \gamma, \sigma; A) > 0$ (C is the constant from (0.6)).

Then the spectrum of the Λ -periodic Dirac operator (0.2) is absolutely continuous.

Remark. For $d = 3$, we can set $M = 4$ (and take the matrices $\widehat{\alpha}_j$ and $\widehat{\beta}$ in the form (0.8)). The condition 3 in Theorem 0.2 admits Coulomb singularities for the functions $\widehat{V}_3^{(s)}$, and therefore for the functions V, V_1 in (0.3), and the function V in (0.4). The condition 3 is fulfilled for the Λ -periodic functions $\widehat{V}_3^{(s)}$ if $\widehat{V}_3^{(s)} \in C^\infty(\mathbb{R}^3 \setminus \bigcup_{j=1}^J (x_j + \Lambda); \mathcal{S}_M^{(s)})$, where x_j , $j = 1, \dots, J$, are different points in K such that $\widehat{V}_3^{(s)}(x) = \widehat{0}$ for x which do not belong to some disjoint closed Λ -periodic neighbourhoods of the sets $x_j + \Lambda$, $j = 1, \dots, J$, and the functions $\widehat{V}_3^{(s)}(\cdot)$ coincide with the functions $|\cdot - x_j|^{-1} \widehat{Q}_j^{(s)}$ in certain neighbourhoods (in \mathbb{R}^3) of points x_j , where $\widehat{Q}_j^{(s)} \in \mathcal{S}_4^{(s)}$ and

$$\|\widehat{Q}_j^{(s)}\|_{\mathcal{M}_4} \leq \min \left\{ \left(\frac{4}{3} \pi \right)^{-1/3} c_1, (2\pi)^{-1} c'_1 \right\}$$

for all $j = 1, \dots, J$ and $s = 0, 1$ (here, the constant c'_1 corresponds to the number $\sigma = 2$).

1 Proof of Theorems 0.1 and 0.2

Let $\widetilde{H}^a(K; \mathbb{C}^M)$, $a > 0$, be the set of vector functions $\varphi : K \rightarrow \mathbb{C}^M$ whose Λ -periodic extensions belong to the Sobolev class $H_{\text{loc}}^a(\mathbb{R}^d; \mathbb{C}^M)$ (of order a); $\widetilde{H}^0(K; \mathbb{C}^M) \doteq L^2(K; \mathbb{C}^M)$. For all $e \in S_{d-1}$, all $k \in \mathbb{R}^d$ ($e_j = (e, \mathcal{E}_j)$, $k_j = (k, \mathcal{E}_j)$, $j = 1, \dots, d$), and all $\varkappa \geq 0$ we introduce the operators

$$\widehat{\mathcal{D}}(k + i\varkappa e) = -i \sum_{j=1}^d \widehat{\alpha}_j \frac{\partial}{\partial x_j} + \sum_{j=1}^d (k_j + i\varkappa e_j) \widehat{\alpha}_j$$

acting on $L^2(K; \mathbb{C}^M)$, with the domain $D(\widehat{\mathcal{D}}(k + i\kappa e)) = \widetilde{H}^1(K; \mathbb{C}^M) \subset L^2(K; \mathbb{C}^M)$.

If $\widehat{W} \in \mathbb{L}_M^\Lambda(d; a)$, $a \geq 0$, then for all $\varepsilon > 0$, all vectors $k \in \mathbb{R}^d$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ we have

$$\begin{aligned} \|\widehat{W}\varphi\| &\leq (a + \varepsilon) \|\widehat{\mathcal{D}}(k)\varphi\| + C_\varepsilon(a, \widehat{W}) \|\varphi\| = \\ &= (a + \varepsilon) \left(\sum_{j=1}^d \left\| \left(k_j - i \frac{\partial}{\partial x_j} \right) \varphi \right\|^2 \right)^{1/2} + C_\varepsilon(a, \widehat{W}) \|\varphi\|, \end{aligned} \quad (1.1)$$

where the operator $\widehat{\mathcal{D}}(k)$ is the operator $\widehat{\mathcal{D}}(k + i\kappa e)$ for $\kappa = 0$, $C_\varepsilon(a, \widehat{W})$ is the constant from (0.5), and the function \widehat{W} is supposed to act on the space $L^2(K; \mathbb{C}^M)$ (here and henceforth, the norms and the scalar products are related to the space $L^2(K; \mathbb{C}^M)$).

Under the conditions of Theorem 0.1, we have

$$\widehat{W} = \widehat{V}^{(0)} + \widehat{V}^{(1)} - \sum_{j=1}^d A_j \widehat{\alpha}_j \in \mathbb{L}_M^\Lambda(d; 0),$$

furthermore, for any $\varepsilon > 0$ there is a constant $C'_\varepsilon(0, \widehat{W}) > 0$ such that for all $k \in \mathbb{R}^d$ and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the estimate

$$\|\widehat{W}\varphi\| \leq \|\widehat{V}^{(0)}\varphi\| + \|\widehat{V}^{(1)}\varphi\| + \|A\|\varphi\| \leq \varepsilon \|\widehat{\mathcal{D}}(k)\varphi\| + C'_\varepsilon(0, \widehat{W}) \|\varphi\| \quad (1.2)$$

holds (we can set $C'_\varepsilon(0, \widehat{W}) = C_{\varepsilon/3}(0, \widehat{V}^{(0)}) + C_{\varepsilon/3}(0, \widehat{V}^{(1)}) + C_{\varepsilon/3}(0, \sum_{j=1}^d A_j \widehat{\alpha}_j)$). If the conditions of Theorem 0.2 are fulfilled, then $\widehat{W} \in \mathbb{L}_M^\Lambda(3; a)$ for some $a \in [0, 1)$.

Suppose $\widehat{W} \in \mathbb{L}_M^\Lambda(d; a)$, $a \in [0, 1)$. Then the operators $\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}$, with the domain $D(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) = \widetilde{H}^1(K; \mathbb{C}^M) \subset L^2(K; \mathbb{C}^M)$, are closed and have compact resolvent for all $k + i\kappa e \in \mathbb{C}^d$. Furthermore, the operators $\widehat{\mathcal{D}}(k) + \widehat{W}$, $k \in \mathbb{R}^d$, are self-adjoint and have a discrete spectrum. Let $\lambda_\nu(k)$, $\nu \in \mathbb{Z}$, be the eigenvalues of the operators $\widehat{\mathcal{D}}(k) + \widehat{W}$. We assume that they are arranged in an increasing order (counting multiplicities). The eigenvalues can be indexed for different $k \in \mathbb{R}^d$ such that the functions $\mathbb{R}^d \ni k \rightarrow \lambda_\nu(k)$ are continuous (see [30]). Denote by

$$K^* = \{y = \sum_{j=1}^d \eta_j E_j^* : 1 \leq \eta_j < 1, j = 1, \dots, d\}$$

the fundamental domain of the lattice Λ^* , $v(K^*)$ is the volume of K^* ; $v(K^*)v(K) = 1$. The periodic Dirac operator (0.1) is unitarily equivalent to the direct integral

$$\int_{2\pi K^*}^\oplus (\widehat{\mathcal{D}}(k) + \widehat{W}) \frac{dk}{(2\pi)^d v(K^*)}. \quad (1.3)$$

The unitary equivalence is established via the Gel'fand transformation (see [44, 11]). The spectrum of the operator (0.1) coincides with $\bigcup_{\nu \in \mathbb{Z}} \{\lambda_\nu(k) : k \in 2\pi K^*\}$.

To prove the absolute continuity of the spectrum of the operator (0.1), it suffices to prove the absence of eigenvalues (of infinite multiplicities) in its spectrum (see [7]). But, on the other hand, if λ is an eigenvalue of the operator (0.1), then the decomposition of the operator (0.1) into the direct integral (1.3) and the analytic Fredholm theorem imply that λ is an eigenvalue of $\widehat{\mathcal{D}}(k+i\kappa e) + \widehat{W}$ for all $k+i\kappa e \in \mathbb{C}^d$ (see [7, 9]). Hence, it suffices to prove that every number $\lambda \in \mathbb{R}$ is not an eigenvalue of the operator $\widehat{\mathcal{D}}(k+i\kappa e) + \widehat{W}$ for some complex vector $k+i\kappa e \in \mathbb{C}^d$ (dependent on λ). This method was used by Thomas in [10]. If, under the conditions of Theorems 0.1 and 0.2, we change $\widehat{W} - \lambda \widehat{I}$ to \widehat{W} (change $\widehat{V}_1^{(0)} - \lambda \widehat{I}$ to $\widehat{V}_1^{(0)}$ in the case of the operator (0.2)), then the new matrix function \widehat{W} satisfies all conditions which are satisfied by the original matrix function \widehat{W} . Therefore, to prove the absolute continuity of the spectrum of the operator (0.1), it suffices to prove the absence of the eigenvalue $\lambda = 0$ in the spectrum of the operator $\widehat{\mathcal{D}}(k+i\kappa e) + \widehat{W}$ for some complex vector $k+i\kappa e \in \mathbb{C}^d$. Thus, Theorems 0.1 and 0.2 are implied by Theorems 1.1 and 1.2, respectively.

Let $\gamma \in \Lambda \setminus \{0\}$ be the vector fixed in Theorems 0.1 and 0.2. In what follows, we denote $e = |\gamma|^{-1}\gamma$.

Theorem 1.1. *Let $d \geq 3$. Suppose the matrix functions $\widehat{V}^{(s)}$, $s = 0, 1$, and the magnetic potential A (for the vector $\gamma \in \Lambda \setminus \{0\}$ and the measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$) satisfy the conditions of Theorem 0.1. Then there exist a number $C_1 > 0$ (we can put $C_1 = \frac{1}{2} \pi |\gamma|^{-1} C_2$, where C_2 is the constant from (1.10)) and a number $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ we have*

$$\|(\widehat{\mathcal{D}}(k+i\kappa e) + \widehat{W})\varphi\| \geq C_1 \|\varphi\|.$$

Theorem 1.2. *Let $d = 3$. Suppose the matrix functions $\widehat{V}^{(s)}$, $s = 0, 1$, and the magnetic potential A (for the vector $\gamma \in \Lambda \setminus \{0\}$ and the measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$) satisfy the conditions of Theorem 0.2 (the constants $c_1 \in (0, C^{-1})$ and $c'_1 > 0$ will be specified later in the course of the proof). Then there exists a number $C'_1 > 0$ (as in Theorem 1.1, we can put $C'_1 = \frac{1}{2} \pi |\gamma|^{-1} C_2$, where C_2 is the constant from (1.10)) and, for every $\Xi > 1$, there exists a number $\kappa_0^{(\Xi)} > 0$ (which also depend on \mathfrak{h} , μ , Λ , γ , σ , on the functions $\widehat{V}_\nu^{(s)}$, $s = 0, 1$, $\nu = 1, 2, 3$, and on the magnetic potential A) such that for any $\Xi > 1$ and any $\kappa_1 \geq \kappa_0^{(\Xi)}$ there is a number $\kappa \in [\kappa_1, \Xi \kappa_1]$ such that for all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the estimate*

$$\|(\widehat{\mathcal{D}}(k+i\kappa e) + \widehat{W})\varphi\| \geq C'_1 \|\varphi\|$$

holds.

Theorems 1.1 and 1.2 are proved in the end of this section. Theorem 1.1 is based on Theorem 1.3. Theorem 1.2 is deduced from Theorems 1.4, 1.5, and 1.6.

For all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$,

$$\widehat{\mathcal{D}}(k+i\kappa e)\varphi = \sum_{N \in \Lambda^*} \widehat{\mathcal{D}}_N(k; \kappa) \varphi_N e^{2\pi i(N, x)},$$

where

$$\widehat{\mathcal{D}}_N(k; \varkappa) = \sum_{j=1}^d (k_j + 2\pi N_j + i\varkappa e_j) \widehat{\alpha}_j, \quad N_j = (N, \mathcal{E}_j), \quad j = 1, \dots, d.$$

For $k \in \mathbb{R}^d$ and $\varkappa \geq 0$, we introduce the notation

$$G_N^\pm(k; \varkappa) \doteq (|k_\parallel + 2\pi N_\parallel|^2 + (\varkappa \pm |k_\perp + 2\pi N_\perp|)^2)^{1/2}, \quad N \in \Lambda^*;$$

$$G_N^+(k; \varkappa) \geq G_N^-(k; \varkappa), \quad G_N^+(k; \varkappa) \geq \varkappa.$$

If $k \in \mathbb{R}^d$ and $|(k, \gamma)| = \pi$, then for all $\varkappa \geq 0$ and all $N \in \Lambda^*$ we have $G_N(k; \varkappa) \geq \pi|\gamma|^{-1}$.

In the case where $|(k, \gamma)| = \pi$, we define the operators \widehat{G}_\pm^ζ , with $\zeta \in \mathbb{C}$, acting on the space $L^2(K; \mathbb{C}^M)$:

$$\begin{aligned} \widehat{G}_\pm^\zeta \varphi &= \sum_{N \in \Lambda^*} (G_N^\pm(k; \varkappa))^\zeta \varphi_N e^{2\pi i(N, x)}, \\ \varphi \in D(\widehat{G}_\pm^\zeta) &= \begin{cases} \widetilde{H}^{\operatorname{Re} \zeta}(K; \mathbb{C}^M) & \text{if } \operatorname{Re} \zeta > 0, \\ L^2(K; \mathbb{C}^M) & \text{if } \operatorname{Re} \zeta \leq 0 \end{cases} \end{aligned}$$

(the operators \widehat{G}_\pm^ζ depend on k and \varkappa , but, in the above notation, this dependence is not indicated explicitly).

For all $k \in \mathbb{R}^d$, all $\varkappa \geq 0$, and all $N \in \Lambda^*$, the inequalities

$$G_N^-(k; \varkappa) \|u\| \leq \|\widehat{\mathcal{D}}_N(k; \varkappa)u\| \leq G_N^+(k; \varkappa) \|u\|, \quad u \in \mathbb{C}^M,$$

hold. Hence, for all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$,

$$\|\widehat{G}_-^1 \varphi\| \leq \|\widehat{\mathcal{D}}(k + i\varkappa e) \varphi\| \leq \|\widehat{G}_+^1 \varphi\|.$$

Let $\widehat{P}^{\mathcal{C}}$, where $\mathcal{C} \subseteq \Lambda^*$, denote the orthogonal projection on $L^2(K; \mathbb{C}^M)$ that takes a vector function $\varphi \in L^2(K; \mathbb{C}^M)$ to the vector function

$$\widehat{P}^{\mathcal{C}} \varphi \doteq \varphi^{\mathcal{C}} = \sum_{N \in \mathcal{C}} \varphi_N e^{2\pi i(N, x)}$$

(in particular, $\widehat{P}^\emptyset = \widehat{0}$). We write $\mathcal{H}(\mathcal{C}) = \{\varphi \in L^2(K; \mathbb{C}^M) : \varphi_N = 0 \text{ for } N \in \Lambda^* \setminus \mathcal{C}\}$.

For vectors $\widetilde{e} \in S_{d-2}(e)$, we define the orthogonal projections on \mathbb{C}^M :

$$\widehat{P}_{\widetilde{e}}^\pm = \frac{1}{2} \left(\widehat{I} \mp i \left(\sum_{j=1}^d e_j \widehat{\alpha}_j \right) \left(\sum_{j=1}^d \widetilde{e}_j \widehat{\alpha}_j \right) \right);$$

$$\|\widehat{P}_{\widetilde{e}'}^\pm \widehat{P}_{\widetilde{e}''}^\mp\| = \|\widehat{P}_{\widetilde{e}'}^\pm - \widehat{P}_{\widetilde{e}''}^\pm\| = \frac{1}{2} |\widetilde{e}'' - \widetilde{e}'|, \quad \widetilde{e}', \widetilde{e}'' \in S_{d-2}(e). \quad (1.4)$$

We shall use the notation $\widetilde{e}(y) \doteq |y_\perp|^{-1} y_\perp \in S_{d-2}(e)$ for vectors $y \in \mathbb{R}^d$ with $y_\perp = y - (y, e)e \neq 0$.

If $k \in \mathbb{R}^d$, $N \in \Lambda^*$, and $k_\perp + 2\pi N_\perp \neq 0$, then

$$\widehat{P}_{\widehat{e}(k+2\pi N)}^\pm \widehat{\mathcal{D}}_N(k; \varkappa) \widehat{P}_{\widehat{e}(k+2\pi N)}^\pm = \widehat{0} \quad (1.5)$$

and, for all vectors $u \in \mathbb{C}^M$ (and all $\varkappa \geq 0$),

$$\|\widehat{\mathcal{D}}_N(k; \varkappa) \widehat{P}_{\widehat{e}(k+2\pi N)}^\pm u\| = G_N^\pm(k; \varkappa) \|\widehat{P}_{\widehat{e}(k+2\pi N)}^\pm u\|. \quad (1.6)$$

If $k_\perp + 2\pi N_\perp = 0$, then

$$G_N^+(k; \varkappa) = G_N^-(k; \varkappa). \quad (1.7)$$

Let denote by $\widehat{P}^\pm = \widehat{P}^\pm(k; e)$ and by $\widehat{P}_*^\pm = \widehat{P}_*^\pm(k; e)$ the orthogonal projections on $L^2(K; \mathbb{C}^M)$:

$$\begin{aligned} \widehat{P}^+ \varphi &= \sum_{N \in \Lambda^* : k_\perp + 2\pi N_\perp \neq 0} \widehat{P}_{\widehat{e}(k+2\pi N)}^+ \varphi_N e^{2\pi i(N, x)}, \\ \widehat{P}_*^+ \varphi &= \widehat{P}^+ \varphi + \sum_{N \in \Lambda^* : k_\perp + 2\pi N_\perp = 0} \varphi_N e^{2\pi i(N, x)}, \\ \widehat{P}_*^- \varphi &= \sum_{N \in \Lambda^* : k_\perp + 2\pi N_\perp \neq 0} \widehat{P}_{\widehat{e}(k+2\pi N)}^- \varphi_N e^{2\pi i(N, x)}, \\ \widehat{P}^- \varphi &= \widehat{P}_*^- \varphi + \sum_{N \in \Lambda^* : k_\perp + 2\pi N_\perp = 0} \varphi_N e^{2\pi i(N, x)}, \quad \varphi \in L^2(K; \mathbb{C}^M) \end{aligned}$$

(the operators \widehat{P}^\pm and \widehat{P}_*^\pm depend on $k \in \mathbb{R}^d$, but the dependence will not be indicated in the notation). Since $\widehat{P}^+ + \widehat{P}^- = \widehat{I}$, $\widehat{P}_*^+ + \widehat{P}_*^- = \widehat{I}$, from equalities (1.5), (1.6), and (1.7) it follows that

$$\|\widehat{P}_*^+ \widehat{\mathcal{D}}(k + i\varkappa e) \varphi\| = \|\widehat{G}_-^1 \widehat{P}^- \varphi\|, \quad \|\widehat{P}_*^- \widehat{\mathcal{D}}(k + i\varkappa e) \varphi\| = \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|, \quad (1.8)$$

$$\|\widehat{\mathcal{D}}(k + i\varkappa e) \varphi\|^2 = \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2$$

for all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$.

Condition **(A₁)** and inequality (0.11) imply that for any $\tau \in (0, 1)$ there is a number $Q = Q(\gamma, A; \tau) > 0$ such that for all $k \in \mathbb{R}^d$ and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the inequality

$$\| |A| \varphi \| \leq \tau \left\| \left(k'_2 - i \frac{\partial}{\partial x'_2} \right) \varphi \right\| + Q \|\varphi\| \quad (1.9)$$

is fulfilled, where $k'_2 = (k, e)$, $x'_2 = (x, e)$, $x \in \mathbb{R}^d$ (we can put

$$Q = \frac{\tau}{4|\gamma|^2} + \frac{8|\gamma|^2}{\tau} \max_{x \in \mathbb{R}^d} \int_0^1 |A(x - \xi \gamma)|^2 d\xi$$

(see (0.11))).

Let $C^*(\mathfrak{h}) > 0$ be the constant defined in Lemma 3.1. The constant $C^*(\mathfrak{h})$ depends on C^* and \mathfrak{h} . (In what follows, we shall assume that $\mathfrak{h} \leq |\gamma|^{-1}$. Therefore the constant

$C^*(\mathfrak{h})$ will depend on the vector γ as well.) Fix a number $\tau \in (0, 1)$ and the corresponding number $Q > 0$. Denote

$$C_2 = (1 - \tau)(1 + Q(1 - \tilde{\theta})^{-1} \frac{|\gamma|}{\pi} e^{\|\mu\| C^*(\mathfrak{h})})^{-1} \in (0, 1). \quad (1.10)$$

Theorem 1.3. *Let $d \geq 3$. Suppose $\widehat{V}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, $A \in L^2(K; \mathbb{R}^d)$ with $A_0 = 0$, $R > 0$, and there are a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, such that for the magnetic potential A , conditions (\mathbf{A}_1) , $(\tilde{\mathbf{A}}_1)$, (\mathbf{A}_2) are satisfied and, moreover, $\widehat{V}_N^{(s)} = \widehat{0}$, $s = 0, 1$, and $A_N = 0$ for all $N \in \Lambda^*$ with $2\pi|N_{\perp}| > R$. Then for any $\delta > 0$ there exist numbers $\tilde{a} = \tilde{a}(C_2; \delta, R) \in (0, C_2]$ and $\varkappa_0 > 0$ such that for all $\varkappa \geq \varkappa_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$ the inequality*

$$\begin{aligned} & \|(\widehat{P}_*^+ + \tilde{a}\widehat{P}_*^-)(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 = \\ & = \|\widehat{P}_*^+(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 + \tilde{a}^2\|\widehat{P}_*^-(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 \geq \\ & \geq (1 - \delta)(C_2^2\|\widehat{G}_-^1\widehat{P}_-^-\varphi\|^2 + \tilde{a}^2\|\widehat{G}_+^1\widehat{P}_+^+\varphi\|^2) \end{aligned}$$

holds.

Theorem 1.3 is proved in Section 2. The following Theorem 1.4, which is proved in Section 4, is a consequence of Theorem 1.3.

Theorem 1.4. *Let $d \geq 3$. Suppose $\widehat{V}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, $A \in L^2(K; \mathbb{R}^d)$ with $A_0 = 0$, and there are a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, such that the magnetic potential A obeys conditions (\mathbf{A}_1) , $(\tilde{\mathbf{A}}_1)$, (\mathbf{A}_2) and, for the functions $\widehat{V}^{(s)}$, $s = 0, 1$, the maps*

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow \widehat{V}^{(s)}(x - \xi\gamma)\} \in L^2([0, 1]; \mathcal{M}_M)$$

are continuous. Then for any $\delta \in (0, 1)$ there exists a number $\varkappa'_0 > 0$ such that for all $\varkappa \geq \varkappa'_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$ the inequality

$$\begin{aligned} & \|(C_2^{-\frac{1}{2}}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}}\widehat{P}_*^-)(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 = \\ & = C_2^{-1}\|\widehat{G}_-^{-\frac{1}{2}}\widehat{P}_*^+(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}}\widehat{P}_*^-(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 \geq \\ & \geq (1 - \delta)(C_2\|\widehat{G}_-^{\frac{1}{2}}\widehat{P}_-^-\varphi\|^2 + \|\widehat{G}_+^{\frac{1}{2}}\widehat{P}_+^+\varphi\|^2) \end{aligned}$$

is fulfilled.

Fix some nonnegative even function $\mathbb{R}^{d-1} \ni x' \rightarrow \mathcal{F}(x') \in \mathbb{R}$ from the Schwartz space $\mathcal{S}(\mathbb{R}^{d-1}; \mathbb{R})$ with the following property: the function

$$\mathbb{R}^{d-1} \ni p \rightarrow \widehat{\mathcal{F}}(p) = \int_{\mathbb{R}^{d-1}} \mathcal{F}(x') e^{i(p, x')} dx'$$

is also nonnegative and satisfies the conditions $\widehat{\mathcal{F}}(0) = 1$ and $\widehat{\mathcal{F}}(p) = 0$ for $|p| \geq 1$ (see, e.g., [43] and remarks after Chapter 1 in [45]). Then we also have $\widehat{\mathcal{F}} \in \mathcal{S}(\mathbb{R}^{d-1}; \mathbb{R})$ and $0 \leq \widehat{\mathcal{F}}(p) = \widehat{\mathcal{F}}(-p) \leq 1$ for all $p \in \mathbb{R}^{d-1}$. Let $\{\mathcal{E}'_j\}$ be an orthogonal basis in \mathbb{R}^d with $\mathcal{E}'_2 = e$. Denote $x'_j = (x, \mathcal{E}'_j)$, $x \in \mathbb{R}^d$, $j = 1, \dots, d$, $x' \doteq (x'_1, x'_3, \dots, x'_d) \in \mathbb{R}^{d-1}$; $x_\perp = \sum_{j \neq 2} x'_j \mathcal{E}'_j \in \mathbb{R}^d$, $x_\parallel = x'_2 e \in \mathbb{R}^d$. For $R > 0$, we set

$$\widehat{V}_{\{R\}}^{(s)}(x) = R^{d-1} \int_{\mathbb{R}^{d-1}} \widehat{V}^{(s)}(x - \sum_{j \neq 2} x'_j \mathcal{E}'_j) \mathcal{F}(Rx'_1, Rx'_3, \dots, Rx'_d) dx', \quad s = 0, 1, \quad (1.11)$$

$$\widehat{V}_{\{R\}}(x) = \widehat{V}_{\{R\}}^{(0)}(x) + \widehat{V}_{\{R\}}^{(1)}(x),$$

$$A_{\{R\}}(x) = R^{d-1} \int_{\mathbb{R}^{d-1}} A(x - \sum_{j \neq 2} x'_j \mathcal{E}'_j) \mathcal{F}(Rx'_1, Rx'_3, \dots, Rx'_d) dx', \quad x \in \mathbb{R}^d. \quad (1.12)$$

Since

$$(\widehat{V}_{\{R\}}^{(s)})_N = \widehat{\mathcal{F}}\left(\frac{2\pi N'}{R}\right) \widehat{V}_N^{(s)}, \quad s = 0, 1, \quad (A_{\{R\}})_N = \widehat{\mathcal{F}}\left(\frac{2\pi N'}{R}\right) A_N,$$

where $N' = (N'_1, N'_3, \dots, N'_d) \in \mathbb{R}^{d-1}$, $N \in \Lambda^*$, we have $(\widehat{V}_{\{R\}}^{(s)})_N = 0$ and $(A_{\{R\}})_N = 0$ for all $N \in \Lambda^*$ with $2\pi|N_\perp| > R$. By the definition of the functions $\widehat{V}_{\{R\}}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$ and $A_{\{R\}} \in L^2(K; \mathbb{R}^d)$ (in the form of a convolution) and by the choice of the function \mathcal{F} , the function $A_{\{R\}}$ (as well as the function A) satisfies conditions (\mathbf{A}_0) , $(\widetilde{\mathbf{A}}_1)$, (\mathbf{A}_2) for all $R > 0$, with the constants $C_\varepsilon(0, \widehat{W}_{\{R\}}) = C_\varepsilon(0, \widehat{W})$, $\varepsilon > 0$ (and inequalities (1.2) are satisfied with the constants $C'_\varepsilon(0, \widehat{W}_{\{R\}}) = C'_\varepsilon(0, \widehat{W})$, $\varepsilon > 0$), moreover, the maps

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow \widehat{V}_{\{R\}}^{(s)}(x - \xi\gamma)\} \in L^2([0, 1]; \mathcal{S}_M^{(s)}), \quad s = 0, 1,$$

$$\mathbb{R}^d \ni x \rightarrow \{[0, 1] \ni \xi \rightarrow A_{\{R\}}(x - \xi\gamma)\} \in L^2([0, 1]; \mathbb{R}^d)$$

are continuous (the function $A_{\{R\}}$ obeys condition (\mathbf{A}_1)), and

$$\| \widehat{W} - \widehat{W}_{\{R\}} \|_{\gamma, M} \rightarrow 0$$

as $R \rightarrow +\infty$. The last relation and inequality (0.11) imply that for any $\widetilde{\varepsilon} > 0$ there is a number $R = R(\widetilde{\varepsilon}) > 0$ (dependent also on γ , μ , and functions $\widehat{V}^{(s)}$, $s = 0, 1$, and A) such that for all $\varepsilon > 0$ and all vector functions $\varphi \in H^1(\mathbb{R}^d; \mathbb{C}^M)$ the inequality

$$\|(\widehat{W} - \widehat{W}_{\{R\}})\varphi\|_{L^2(\mathbb{R}^d; \mathbb{C}^M)} \leq \widetilde{\varepsilon} \left(\varepsilon \left\| -i \frac{\partial \varphi}{\partial x'_2} \right\|_{L^2(\mathbb{R}^d; \mathbb{C}^M)} + C'(\gamma, \varepsilon) \|\varphi\|_{L^2(\mathbb{R}^d; \mathbb{C}^M)} \right) \quad (1.13)$$

holds, where $x'_2 = (x, e)$, $x \in \mathbb{R}^d$, and $C'(\gamma, \varepsilon)$ is the constant from inequality (0.11). From (1.13) it follows that

$$\|(\widehat{W} - \widehat{W}_{\{R\}})\varphi\| \leq \widetilde{\varepsilon} \left(\varepsilon \left\| (k'_2 - i \frac{\partial}{\partial x'_2}) \varphi \right\| + C'(\gamma, \varepsilon) \|\varphi\| \right) \quad (1.14)$$

for all $\tilde{\varepsilon} > 0$ ($R = R(\tilde{\varepsilon})$), all $\varepsilon > 0$, all $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$, and all $k \in \mathbb{R}^d$, where $k'_2 = (k, e)$.

Proof of Theorem 1.1. Given the vector $\gamma \in \Lambda \setminus \{0\}$ and the measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, suppose the functions $\widehat{V}^{(s)}$, $s = 0, 1$, and A satisfy the conditions of Theorem 0.1. For the number

$$\tilde{\varepsilon} \doteq \frac{1}{4\sqrt{3}} C_2 \left(1 + \frac{|\gamma|^2}{\pi^2} (C'(\gamma, 1))^2\right)^{-1/2},$$

there is a number $R = R(\tilde{\varepsilon}) > 0$ such that inequality (1.14) holds for all $\varepsilon > 0$, all vectors $k \in \mathbb{R}^d$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$. Furthermore, $(\widehat{W}_{\{R\}})_N = \widehat{0}$ for all $N \in \Lambda^*$ with $2\pi|N_{\perp}| > R$, the function $A_{\{R\}}$ obeys conditions (\mathbf{A}_0) , (\mathbf{A}_1) , $(\tilde{\mathbf{A}}_1)$, (\mathbf{A}_2) (with the vector $\gamma \in \Lambda \setminus \{0\}$, the measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, and the constants C^* , $\tilde{\theta}$), and for the function $A_{\{R\}}$, estimates (1.9) are fulfilled with the constants $Q(\gamma, A_{\{R\}}; \tau) = Q(\gamma, A; \tau)$, $\tau \in (0, 1)$ (including the chosen number τ). Therefore, Theorem 1.3 applied to the functions $\widehat{V}_{\{R\}}^{(s)}$, $s = 0, 1$, $A_{\{R\}}$ and the number $\delta = \frac{1}{3}$, implies that there exist a number $\tilde{a} = \tilde{a}(C_2; \frac{1}{3}, R(\tilde{\varepsilon})) \in (0, C_2]$ and a number $\varkappa_0 > 0$ such that the inequality

$$\begin{aligned} \|(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}})\varphi\|^2 &\geq \|(\widehat{P}_*^+ + \tilde{a}\widehat{P}_*^-)(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}})\varphi\|^2 \geq \\ &\geq \frac{2}{3} (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \tilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2) \end{aligned}$$

holds for all $\varkappa \geq \varkappa_0$, all vectors $k \in \mathbb{R}^d$, with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$. Set $\varepsilon = \tilde{a}(4\sqrt{6}\tilde{\varepsilon})^{-1}$ and assume that $\tilde{a}\varkappa_0 \geq \pi|\gamma|^{-1}C_2$ and $\varepsilon\varkappa_0 \geq C'(\gamma, \varepsilon)$. Then, for $\varkappa \geq \varkappa_0$, from (1.14) it follows that

$$\begin{aligned} \|(\widehat{W} - \widehat{W}_{\{R\}})\widehat{P}^- \varphi\|^2 &\leq 2\tilde{\varepsilon}^2 (\|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + (C'(\gamma, 1))^2 \|\widehat{P}^- \varphi\|^2) \leq \frac{1}{24} C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2, \\ \|(\widehat{W} - \widehat{W}_{\{R\}})\widehat{P}^+ \varphi\|^2 &\leq 2\tilde{\varepsilon}^2 (\varepsilon^2 \|\widehat{G}_-^1 \widehat{P}^+ \varphi\|^2 + (C'(\gamma, \varepsilon))^2 \|\widehat{P}^+ \varphi\|^2) \leq \\ &\leq 2\tilde{\varepsilon}^2 (\varepsilon^2 + \varkappa^{-2} (C'(\gamma, \varepsilon))^2) \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2 \leq \frac{1}{24} \tilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2. \end{aligned}$$

Hence, for $\varkappa \geq \varkappa_0$ (and for all $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$) we have

$$\begin{aligned} &\|(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 \geq \\ &\geq \frac{1}{2} \|(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}})\varphi\|^2 - 2 \|(\widehat{W} - \widehat{W}_{\{R\}})\widehat{P}^- \varphi\|^2 - 2 \|(\widehat{W} - \widehat{W}_{\{R\}})\widehat{P}^+ \varphi\|^2 \geq \\ &\geq \frac{1}{4} (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \tilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2) \geq \\ &\geq \frac{1}{4} (C_2^2 \frac{\pi^2}{|\gamma|^2} \|\widehat{P}^- \varphi\|^2 + \tilde{a}^2 \varkappa^2 \|\widehat{P}^+ \varphi\|^2) \geq C_1^2 \|\varphi\|^2, \end{aligned}$$

where $C_1 = \frac{1}{2} \pi |\gamma|^{-1} C_2$. □

Remark. Theorem 1.1 can be also proved using Theorem 1.4 (see the proof of Theorem 1.2).

Theorem 1.5. Let $d = 3$, $\gamma \in \Lambda \setminus \{0\}$. Suppose $\widehat{V}^{(s)} \in L^3 \ln^{1+\delta} L(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, for some $\delta > 0$. Then for any $\varepsilon' > 0$ and any $\Xi > 1$ there is a number $\varkappa_0 = \varkappa_0(\varepsilon', \Xi) > 0$ (dependent also on γ and the functions $\widehat{V}^{(s)}$) such that for every $\varkappa_1 \geq \varkappa_0$ there exists a number $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$ such that for all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the inequality

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ \widehat{V} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^- \widehat{V} \varphi\|^2 \leq (\varepsilon')^2 (\|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2)$$

holds, where $\widehat{V} = \widehat{V}^{(0)} + \widehat{V}^{(1)}$.

Theorem 1.6. Let $d = 3$, $\gamma \in \Lambda \setminus \{0\}$, $\sigma \in (0, 2]$. Then there exist a universal constant $\widetilde{c}_1 \in (0, C^{-1})$ and a constant $\widetilde{c}'_1 > 0$, dependent only on σ , such that for any $\varepsilon' > 0$ and for all matrix functions

$$\widehat{V}^{(s)} = \sum_{q=1}^{Q_s} \widehat{V}_q^{(s)}, \quad s = 0, 1,$$

that satisfy (for $s = 0$ and $s = 1$) the following conditions:

- 1) $\widehat{V}_q^{(s)} \in L_w^3(K; \mathcal{S}_M^{(s)})$ and $\beta_{\gamma, \sigma}(0; \widehat{V}_q^{(s)}) < +\infty$, $q = 1, \dots, Q_s$,
- 2) the essential supports $\text{supp } \widehat{V}_q^{(s)}$ of functions $\widehat{V}_q^{(s)}$ (assumed to be Λ -periodic functions defined on space \mathbb{R}^3) do not intersect for different q ,
- 3) $\|\widehat{V}^{(s)}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} = \max_{q=1, \dots, Q_s} \|\widehat{V}_q^{(s)}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \leq \widetilde{c}_1 \varepsilon'$,
- 4) $\max_{q=1, \dots, Q_s} \beta_{\gamma, \sigma}(\widehat{V}_q^{(s)}) \leq \widetilde{c}'_1 \varepsilon'$,

there exists a number $\varkappa_0''' > 0$ such that for all $\varkappa \geq \varkappa_0'''$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the inequality

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ \widehat{V} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^- \widehat{V} \varphi\|^2 \leq (\varepsilon')^2 (\|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2) \quad (1.15)$$

holds (where $\widehat{V} = \widehat{V}^{(0)} + \widehat{V}^{(1)}$).

Theorems 1.5 and 1.6 are proved in Sections 5 and 6, respectively.

Proof of Theorem 1.2. Suppose the functions $\widehat{V}_\nu^{(s)}$, $s = 0, 1$, $\nu = 1, 2, 3$, and A satisfy the conditions of Theorem 0.2, and the constants $c_1 \in (0, C^{-1})$ and $c'_1 > 0$ are chosen in accordance with Theorem 1.6 (see the conditions of Theorem 0.2 for the functions $\widehat{V}_3^{(s)}$): $c_1 = \widetilde{c}_1 \varepsilon'$, $c'_1 = \widetilde{c}'_1 \varepsilon'$, here $\varepsilon' \doteq \frac{1}{4\sqrt{3}} C_2$ (the constant $C_2 \in (0, 1)$ (see (1.10)) is determined by the magnetic potential A and by the choice of the vector $\gamma \in \Lambda \setminus \{0\}$ and the measure $\mu \in \mathfrak{M}_h$, $h > 0$; the choice of the vector γ depends also on matrix functions $\widehat{V}_1^{(s)}$, $s = 0, 1$). Denote

$$\widehat{W}_1 = \widehat{V}_1 - \sum_{j=1}^3 A_j \widehat{\alpha}_j.$$

Given $\delta = \frac{1}{3}$, from Theorem 1.4 it follows that there is a number $\varkappa'_0 > 0$ such that for all $\varkappa \geq \varkappa'_0$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the

following inequality is valid:

$$\begin{aligned} & \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_1) \varphi \|^2 \geqslant \\ & \geqslant \frac{2}{3} (C_2 \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2 + \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2). \end{aligned} \quad (1.16)$$

Fix a number $\Xi > 1$. From Theorems 1.5 and 1.6 (applied to the matrix functions $\widehat{V}_2^{(s)}$ and $\widehat{V}_3^{(s)}$, respectively) it follows that there exists a number $\kappa_0^{(\Xi)} \geqslant \max\{\kappa'_0, \pi|\gamma|^{-1}C_2\}$ such that for any $\kappa_1 \geqslant \kappa_0^{(\Xi)}$ there is a number $\kappa \in [\kappa_1, \Xi\kappa_1]$ such that for all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ we have

$$\| (\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) \widehat{V}_\nu \varphi \|^2 \leqslant (\varepsilon')^2 (\|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2), \quad \nu = 2, 3, \quad (1.17)$$

where $\widehat{V}_\nu = \widehat{V}_\nu^{(0)} + \widehat{V}_\nu^{(1)}$. Now, inequalities (1.16) and (1.17) (and also the relations $C_2 \in (0, 1)$, $G_N^+(k; \kappa) \geqslant \kappa$ and $G_N^-(k; \kappa) \leqslant G_N^+(k; \kappa)$, $N \in \Lambda^*$) imply that for every $\kappa_1 \geqslant \kappa_0^{(\Xi)} \geqslant \pi|\gamma|^{-1}C_2$ and for the number $\kappa \in [\kappa_1, \Xi\kappa_1]$ chosen as above, the following estimates are hold for all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$:

$$\begin{aligned} & C_2^{-1} \frac{|\gamma|}{\pi} \| (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 \geqslant \\ & \geqslant \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 \geqslant \\ & \geqslant \frac{1}{2} \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_1) \varphi \|^2 - \\ & \quad - 2 \sum_{\nu=2}^3 \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) \widehat{V}_\nu \varphi \|^2 \geqslant \\ & \geqslant \frac{1}{3} (C_2 \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2 + \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2) - 2C_2^{-1} \sum_{\nu=2}^3 \| (\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) \widehat{V}_\nu \varphi \|^2 \geqslant \\ & \geqslant \frac{1}{4} (C_2 \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2 + \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2) \geqslant \\ & \geqslant \frac{1}{4} (C_2 \frac{\pi}{|\gamma|} \|\widehat{P}^- \varphi\|^2 + \kappa \|\widehat{P}^+ \varphi\|^2) \geqslant C_2^{-1} \frac{|\gamma|}{\pi} C_1^2 \|\varphi\|^2, \end{aligned}$$

where $C_1 = \frac{1}{2} \pi |\gamma|^{-1} C_2$. □

2 Proof of Theorem 1.3

Given the vector $\gamma \in \Lambda \setminus \{0\}$ ($e = |\gamma|^{-1} \gamma$), for all $\varepsilon \in (0, 1)$, define the sets

$$\mathcal{C}(\varepsilon) = \mathcal{C}(k, \kappa; \varepsilon) = \{N \in \Lambda^* : |\kappa - |k_\perp + 2\pi N_\perp|| < \varepsilon \kappa\},$$

$k \in \mathbb{R}^d$, $\kappa > 0$.

In this Section, Theorem 1.3 is deduced from Theorem 2.1 which is a weakened variant of Theorem 1.3. Theorem 2.1 is proved in Section 3.

Theorem 2.1. Let $d \geq 3$. Suppose $\widehat{V}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, $A \in L^2(K; \mathbb{R}^d)$ with $A_0 = 0$, $R > 0$, and there are a vector $\gamma \in \Lambda \setminus \{0\}$ and a measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, such that for the magnetic potential A , conditions (\mathbf{A}_1) , $(\widetilde{\mathbf{A}}_1)$, (\mathbf{A}_2) are fulfilled, and, moreover, $\widehat{V}_N^{(s)} = \widehat{0}$, $s = 0, 1$, and $A_N = 0$ for all $N \in \Lambda^*$ with $2\pi|N_{\perp}| > R$. Then for any $\delta \in (0, 1)$ there exist numbers $a = a(C_2; \delta, R) \in (0, C_2]$ and $\varkappa_0 > 2R$ (the number \varkappa_0 depends on δ , $|\gamma|$, \mathfrak{h} , $\|\mu\|$, R and on the constants $C'_\varepsilon(0, \widehat{W})$, C^* , τ , Q , $\widetilde{\theta}$) such that for all $\varkappa \geq \varkappa_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$ the inequality

$$\begin{aligned} & \|\widehat{P}^+(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 + a^2 \|\widehat{P}^-(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 \geq \\ & \geq (1 - \delta) (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + a^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2), \end{aligned} \quad (2.1)$$

holds, where the constant C_2 is defined in (1.10).

Remark. Since $\varkappa_0 > 2R$, we see that for all $\varkappa \geq \varkappa_0$ and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$ the equality $(\widehat{W}\varphi)_N = 0$ holds for $N \in \Lambda^*$ with $|k_{\perp} + 2\pi N_{\perp}| \leq \frac{\varkappa}{2} - R$ (in particular, $(\widehat{W}\varphi)_N = 0$ for $N \in \Lambda^*$ with $k_{\perp} + 2\pi N_{\perp} = 0$). Hence, in the left hand side of inequality (2.1), the orthogonal projections \widehat{P}^{\pm} may be replaced by the orthogonal projections \widehat{P}_*^{\pm} .

Lemma 2.1. Under the conditions of Theorem 1.2, for any $\varepsilon > 0$ there is a number $\widetilde{\varkappa}_0 = \widetilde{\varkappa}_0(\varepsilon) > 0$ such that for all $\varkappa \geq \widetilde{\varkappa}_0$, all vectors $k \in \mathbb{R}^d$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\Lambda^* \setminus \mathcal{C}(\frac{1}{4}))$ the estimate

$$\|\widehat{W}\varphi\| \leq \varepsilon \|\widehat{G}_-^1 \varphi\|$$

holds.

Proof. Indeed, set $\widetilde{\varkappa}_0 = 8\varepsilon^{-1}C_{\varepsilon/10}(0, \widehat{W})$ (here $C_{\varepsilon/10}(0, \widehat{W})$ is the constant from (0.5)). Since $G_N^-(k; \varkappa) \geq \frac{\varkappa}{4}$ and $G_N^-(k; \varkappa) \geq \frac{1}{5}|k + 2\pi N|$ for all $N \in \Lambda^* \setminus \mathcal{C}(\frac{1}{4})$, we obtain (see (1.1))

$$\begin{aligned} \|\widehat{W}\varphi\| & \leq \frac{\varepsilon}{10} \left\| \sum_{j=1}^d \left(k_j - i \frac{\partial}{\partial x_j} \right) \widehat{\alpha}_j \varphi \right\| + C_{\varepsilon/10}(0, \widehat{W}) \|\varphi\| = \\ & = \frac{\varepsilon}{10} v^{1/2}(K) \left(\sum_{N \in \Lambda^* \setminus \mathcal{C}(\frac{1}{4})} |k + 2\pi N|^2 \|\varphi_N\|^2 \right)^{1/2} + C_{\varepsilon/10}(0, \widehat{W}) \|\varphi\| \leq \\ & \leq \left(\frac{\varepsilon}{2} + \frac{4}{\varkappa} C_{\varepsilon/10}(0, \widehat{W}) \right) \|\widehat{G}_-^1 \varphi\| \leq \varepsilon \|\widehat{G}_-^1 \varphi\| \end{aligned}$$

for all $\varkappa \geq \widetilde{\varkappa}_0$ (all $k \in \mathbb{R}^d$ and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\Lambda^* \setminus \mathcal{C}(\frac{1}{4}))$). \square

Proof of Theorem 1.3. We write $\delta_1 = \frac{\delta}{3}$. Let $\widetilde{a} = a(C_2; \delta_1, R) \in (0, C_2]$ and \varkappa_0 be the numbers defined in Theorem 2.1. Denote

$$\varepsilon = \frac{\delta_1}{\sqrt{6(1 - \delta_1)}} \min \{C_2, \widetilde{a}\}.$$

Without loss of generality we assume that $\varkappa_0 \geq 4R$ and $\varkappa_0 \geq \tilde{\varkappa}_0(\varepsilon)$, where $\tilde{\varkappa}_0(\varepsilon)$ is the number from Lemma 2.1. In what follows, we also assume that $\varkappa \geq \varkappa_0$ and the vector $k \in \mathbb{R}^d$ satisfies the equality $|(k, \gamma)| = \pi$. For the vector function $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$, the equality

$$\begin{aligned} & \|(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi\|^2 = \\ & = \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi\|^2 + \\ & + \|\hat{P}^{\Lambda^* \setminus c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi\|^2 \end{aligned} \quad (2.2)$$

holds. We shall estimate the summands in the right hand side of (2.2). Since $\varkappa \geq \varkappa_0 \geq 4R$ and $\widehat{W}_N = 0$ for all $N \in \Lambda^*$ with $2\pi|N_\perp| > R$, we see that $\widehat{W}_N = 0$ for all $N \in \Lambda^*$ with $2\pi|N_\perp| > \frac{\varkappa}{4}$. The last assertion will be used below to obtain necessary estimates. We have

$$\begin{aligned} & \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi\| = \\ & = \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})(\varphi^{c(\frac{1}{2})} + \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})})\| \geq \\ & \geq \|(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{1}{2})}\| - \\ & - \|\hat{P}^{\Lambda^* \setminus c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{1}{2})}\| - \\ & - \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\|. \end{aligned} \quad (2.3)$$

Using Lemma 2.1, we get

$$\begin{aligned} & \|\hat{P}^{\Lambda^* \setminus c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{1}{2})}\| = \\ & = \|\hat{P}^{\Lambda^* \setminus c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)\widehat{W}\varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\| \leq \|\widehat{W}\varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\| \leq \varepsilon \|\widehat{G}_-^1 \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\|, \end{aligned} \quad (2.4)$$

$$\begin{aligned} & \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\| = \\ & = \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)\widehat{W}\varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\| \leq \|\widehat{W}\varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\| \leq \varepsilon \|\widehat{G}_-^1 \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\|. \end{aligned} \quad (2.5)$$

On the other hand, from Theorem 2.1 we derive

$$\begin{aligned} & \|(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{1}{2})}\|^2 \geq \\ & \geq (1 - \delta_1) (C_2^2 \|\widehat{G}_-^1 \hat{P}^- \varphi^{c(\frac{1}{2})}\|^2 + \tilde{a}^2 \|\widehat{G}_+^1 \hat{P}^+ \varphi^{c(\frac{1}{2})}\|^2). \end{aligned}$$

Consequently, from (2.3), (2.4), and (2.5) we obtain

$$\begin{aligned} & \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi\|^2 \geq \\ & \geq (1 - \delta_1) \|(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{1}{2})}\|^2 - \\ & - \frac{2(1 - \delta_1)}{\delta_1} \|\hat{P}^{\Lambda^* \setminus c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{1}{2})}\|^2 - \\ & - \frac{2(1 - \delta_1)}{\delta_1} \|\hat{P}^{c(\frac{1}{2})}(\hat{P}^+ + \tilde{a}\hat{P}^-)(\hat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\|^2 \geq \end{aligned} \quad (2.6)$$

$$\begin{aligned}
&\geq (1 - \delta_1)^2 (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi^{c(\frac{1}{2})}\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi^{c(\frac{1}{2})}\|^2) - \\
&- \frac{2(1 - \delta_1)}{\delta_1} \varepsilon^2 (\|\widehat{G}_-^1 \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\|^2 + \|\widehat{G}_-^1 \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\|^2) \geq \\
&\geq (1 - 2\delta_1) (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi^{c(\frac{1}{2})}\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi^{c(\frac{1}{2})}\|^2) - \\
&- \frac{2(1 - \delta_1)}{\delta_1} \varepsilon^2 \|\widehat{G}_-^1 \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{4})}\|^2.
\end{aligned}$$

Let us estimate the second summand in the right hand side of (2.2). By Lemma 2.1, we have

$$\begin{aligned}
&\|\widehat{P}^{\Lambda^* \setminus c(\frac{1}{2})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi\| = \\
&= \|\widehat{P}^{\Lambda^* \setminus c(\frac{1}{2})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) (\varphi^{\Lambda^* \setminus c(\frac{1}{2})} + \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})})\| \geq \\
&\geq \|(\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| - \\
&- \|\widehat{P}^{c(\frac{1}{2}) \setminus c(\frac{1}{4})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| - \\
&- \|\widehat{P}^{\Lambda^* \setminus c(\frac{1}{2})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\| = \\
&= \|(\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| - \\
&- \|\widehat{P}^{c(\frac{1}{2}) \setminus c(\frac{1}{4})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-) \widehat{W} \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\| - \\
&- \|\widehat{P}^{\Lambda^* \setminus c(\frac{1}{2})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-) \widehat{W} \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\| \geq \\
&\geq \|(\widehat{P}^+ + \widetilde{a}\widehat{P}^-) \widehat{\mathcal{D}}(k + i\kappa e) \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| - \\
&- \|\widehat{W} \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| - \|\widehat{W} \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\| - \|\widehat{W} \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\| \geq \\
&\geq \|(\widehat{P}^+ + \widetilde{a}\widehat{P}^-) \widehat{\mathcal{D}}(k + i\kappa e) \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| - \\
&- \varepsilon (\|\widehat{G}_-^1 \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\| + \|\widehat{G}_-^1 \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\| + \|\widehat{G}_-^1 \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\|).
\end{aligned}$$

Therefore (taking into account inequalities (1.8) and the choice of the number $C_2 \in (0, 1)$),

$$\begin{aligned}
&\|\widehat{P}^{\Lambda^* \setminus c(\frac{1}{2})} (\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi\|^2 \geq \\
&\geq (1 - 2\delta_1) \|(\widehat{P}^+ + \widetilde{a}\widehat{P}^-) \widehat{\mathcal{D}}(k + i\kappa e) \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2 - \\
&- \frac{3(1 - 2\delta_1)}{2\delta_1} \varepsilon^2 \|\widehat{G}_-^1 \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2 - \\
&- \frac{3(1 - 2\delta_1)}{2\delta_1} \varepsilon^2 (\|\widehat{G}_-^1 \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{2})}\|^2 + \|\widehat{G}_-^1 \varphi^{c(\frac{1}{2}) \setminus c(\frac{1}{4})}\|^2) = \\
&= (1 - 2\delta_1) (\|\widehat{G}_-^1 \widehat{P}^- \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2) - \\
&- \frac{3(1 - 2\delta_1)}{2\delta_1} \varepsilon^2 (\|\widehat{G}_-^1 \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2 + \|\widehat{G}_-^1 \varphi^{c(\frac{3}{4}) \setminus c(\frac{1}{4})}\|^2) \geq \\
&\geq (1 - 2\delta_1) (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi^{\Lambda^* \setminus c(\frac{1}{2})}\|^2) -
\end{aligned}$$

$$- \frac{2(1-\delta_1)}{\delta_1} \varepsilon^2 \left(\|\widehat{G}_-^1 \varphi^{\Lambda^* \setminus \mathcal{C}(\frac{1}{2})}\|^2 + \|\widehat{G}_-^1 \varphi^{\mathcal{C}(\frac{3}{4}) \setminus \mathcal{C}(\frac{1}{4})}\|^2 \right).$$

From the last inequality and from (2.2), (2.6), it follows that the estimate

$$\begin{aligned} & \|(\widehat{P}^+ + \widetilde{a}\widehat{P}^-)(\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W})\varphi\|^2 \geq \\ & \geq (1 - 2\delta_1) \left(C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2 \right) - \frac{6(1-\delta_1)}{\delta_1} \varepsilon^2 \|\widehat{G}_-^1 \varphi\|^2 \end{aligned} \quad (2.7)$$

holds. Finally, estimate (2.7) and the estimate

$$\begin{aligned} & \frac{6(1-\delta_1)}{\delta_1} \varepsilon^2 \|\widehat{G}_-^1 \varphi\|^2 = \frac{6(1-\delta_1)}{\delta_1} \varepsilon^2 \left(\|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \|\widehat{G}_-^1 \widehat{P}^+ \varphi\|^2 \right) \leq \\ & \leq \delta_1 \left(C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2 \right) \leq \delta_1 \left(C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \widetilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2 \right) \end{aligned}$$

imply inequality (2.1). This completes the proof.

3 Proof of Theorem 2.1

First we shall show that it suffices to assume that the magnetic potential A is a trigonometric polynomial. Indeed, suppose the functions $\widehat{V}^{(s)}$, $s = 0, 1$, and A satisfy the conditions of Theorem 2.1 (in particular, for the vector $\gamma \in \Lambda \setminus \{0\}$ and the measure $\mu \in \mathfrak{M}_{\mathfrak{h}}$, $\mathfrak{h} > 0$, conditions (\mathbf{A}_1) , $(\widetilde{\mathbf{A}}_1)$, (\mathbf{A}_2) are fulfilled, moreover, $A_0 = 0$ and $\widehat{V}_N^{(s)} = \widehat{0}$, $s = 0, 1$, $A_N = 0$ for all $N \in \Lambda^*$ with $2\pi|N_{\perp}| > R$). Let $\mathcal{G} : \mathbb{R}^d \rightarrow \mathbb{R}$ be a nonnegative function from the Schwartz space $\mathcal{S}(\mathbb{R}^d; \mathbb{R})$ such that for the function

$$\mathbb{R}^d \ni p \rightarrow \widehat{\mathcal{G}}(p) = \int_{\mathbb{R}^d} \mathcal{G}(x) e^{i(p,x)} dx$$

we have $\widehat{\mathcal{G}}(0) = 1$ and $\widehat{\mathcal{G}}(p) = 0$ for $|p| \geq 1$ (we may assume that the function \mathcal{G} coincides with the function $\mathcal{F} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ considered in Section 1 if we change \mathbb{R}^{d-1} to \mathbb{R}^d). Let us denote

$$A^{\{r\}}(x) = r^d \int_{\mathbb{R}^d} A(x-y) \mathcal{G}(ry) dy, \quad r > 0, \quad x \in \mathbb{R}^d.$$

For any $r > 0$, the functions $\widehat{V}^{(s)}$ and $A^{\{r\}}$ (as well as the functions $\widehat{V}^{(s)}$ and A) satisfy all conditions of Theorem 2.1 (with the vector γ , the measure μ and the constants $C'_\varepsilon(0, \widehat{V} - \sum_{j=1}^d A_j^{\{r\}} \widehat{\alpha}_j) = C'_\varepsilon(0, \widehat{W})$, C^* , τ , Q , $\widetilde{\theta}$). Furthermore,

$$(A^{\{r\}})_N = \widehat{\mathcal{G}}\left(-\frac{2\pi N}{r}\right) A_N, \quad N \in \Lambda^*,$$

hence, $(A^{\{r\}})_N = 0$ for $2\pi|N| \geq r$. Now, if we suppose that Theorem 2.1 is already proved for the matrix functions $\widehat{V}^{(s)}$, $s = 0, 1$, and the magnetic potentials $A^{\{r\}}$ (which are trigonometric polynomials), $r > 0$, then for any $\delta \in (0, 1)$ there exist numbers $\varkappa_0 > 2R$

and $a \in (0, 1)$ (independent of the number $r > 0$) such that for all $\varkappa \geq \varkappa_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$ the inequality

$$\|(\hat{P}^+ + a\hat{P}^-)(\hat{\mathcal{D}}(k + i\varkappa e) + \hat{V} - \sum_{j=1}^d A_j^{\{r\}} \hat{\alpha}_j) \phi\|^2 \geq \quad (3.1)$$

$$\geq (1 - \delta) (C_2^2 \|\hat{G}_-^1 \hat{P}^- \varphi\|^2 + a^2 \|\hat{G}_+^1 \hat{P}^+ \varphi\|^2)$$

holds. On the other hand, $A \in L^2(K; \mathbb{R}^d)$ and $\|A - A^{\{r\}}\|_{L^2(K; \mathbb{R}^d)} \rightarrow 0$ as $r \rightarrow +\infty$. Hence, assuming that the vector function φ is a trigonometric polynomial and taking the limit in (3.1) as $r \rightarrow +\infty$, we obtain inequality (2.1). Since trigonometric polynomials from the set $\mathcal{H}(\mathcal{C}(\frac{1}{2}))$ are dense in $\tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$ (with respect to the norm of the space $\tilde{H}^1(K; \mathbb{C}^M)$) and the magnetic potential A obeys condition (\mathbf{A}_1) (therefore, $\sum_{j=1}^d A_j \hat{\alpha}_j \in \mathbb{L}_M^\Lambda(d; 0)$), we see that inequality (2.1) holds for all $\varkappa \geq \varkappa_0$, all $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$. Thus, without loss of generality we shall assume that the magnetic potential $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a trigonometric polynomial.

Since the number $\mathfrak{h} > 0$ can be chosen arbitrarily small, we shall also assume that $\mathfrak{h} \leq |\gamma|^{-1}$.

In this Section, we use the method suggested in [46] (also see [42]). Lower bounds for the number $\varkappa_0 > 2R$ are specified in the course of the proof. Adding new lower bounds, we assume that all previous bounds are valid as well. Let $\delta \in (0, 1)$. We write $\delta_1 = \frac{\delta}{8}$, $\delta_2 = \frac{\delta}{4}$. Let us denote

$$C_3 = 1 + \tau + \frac{|\gamma|}{\pi} Q.$$

Suppose a number $\tilde{\varepsilon} \in (0, \frac{\delta}{8}]$ satisfies the inequality

$$\tilde{\varepsilon} C_3^2 < \frac{1}{400} \delta^2 (1 - \tilde{\varepsilon}) C_2^2.$$

If $\varkappa \geq \varkappa_0 > 2R$, $k \in \mathbb{R}^d$, $N \in \mathcal{C}(\frac{1}{2})$ and $N' \in \Lambda^*$ with $2\pi|N'_\perp| \leq R$, then $|k_\perp + 2\pi(N_\perp + N'_\perp)| > \frac{\varkappa}{2} - R > 0$ and

$$|\tilde{e}(k + 2\pi(N + N')) - \tilde{e}(k + 2\pi N)| < \frac{2R}{\varkappa}. \quad (3.2)$$

There are numbers $\tilde{c} = \tilde{c}(\tilde{\varepsilon}) > 0$ and $\varkappa'_0 > (\tilde{c} + 4)R$ such that for all $\varkappa \geq \varkappa_0 \geq \varkappa'_0$, there are nonintersecting (nonempty) open (in $S_{d-2}(e)$) sets $\tilde{\Omega}_\lambda \subset S_{d-2}(e)$ and vectors $\tilde{e}^\lambda \in \tilde{\Omega}_\lambda$, $\lambda \in \mathcal{L} = \{1, \dots, \lambda_0(d, \tilde{\varepsilon}, R; \varkappa)\}$, such that

1) $|\tilde{e} - \tilde{e}^\lambda| < \tilde{\rho} = \tilde{c} \frac{R}{\varkappa}$ for all $\tilde{e} \in \tilde{\Omega}_\lambda$;

2) $|\tilde{e}' - \tilde{e}''| > \frac{8R}{\varkappa}$ for all $\tilde{e}' \in \tilde{\Omega}_{\lambda_1}$, $\tilde{e}'' \in \tilde{\Omega}_{\lambda_2}$, $\lambda_1 \neq \lambda_2$;

3a) $\text{meas } S_{d-2}(e) \setminus \bigcup_\lambda \tilde{\Omega}_\lambda < \frac{1}{2} \tilde{\varepsilon} \text{meas } S_{d-2}(e)$, where *meas* stands for the (invariant) surface measure on the $(d - 2)$ -dimensional sphere $S_{d-2}(e)$.

The sets $\tilde{\Omega}_\lambda$ and the vectors \tilde{e}^λ may be substituted by the sets $\hat{\Theta}\tilde{\Omega}_\lambda$ and the vectors $\hat{\Theta}\tilde{e}^\lambda$ (which also have properties 1, 2, and 3a), where $\hat{\Theta}$ is any orthogonal transformation of the subspace $\{x \in \mathbb{R}^d : (x, \gamma) = 0\}$. Therefore, choosing an appropriate orthogonal transformation $\hat{\Theta}$ of the subspace $\{x \in \mathbb{R}^d : (x, \gamma) = 0\}$ and using the notation $\tilde{\Omega}_\lambda$ and \tilde{e}^λ instead of $\hat{\Theta}\tilde{\Omega}_\lambda$ and $\hat{\Theta}\tilde{e}^\lambda$ respectively, we can also assume that for the vector $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and for the vector function $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$, the following property is fulfilled (for both signs $+$ and $-$):

3b)

$$\sum_{N \in \mathcal{C}(\frac{1}{2}) : \tilde{e}(k+2\pi N) \notin \bigcup_{\lambda} \tilde{\Omega}_\lambda} \|\hat{G}_N^\pm(k; \varkappa) \hat{P}_{\tilde{e}(k+2\pi N)}^\pm \varphi_N\|^2 \leq \tilde{\varepsilon} v^{-1}(K) \|\hat{G}_\pm^1 \hat{P}^\pm \varphi\|^2.$$

The sets $\tilde{\Omega}_\lambda$ and the vectors \tilde{e}^λ depend on $d, \gamma, \tilde{\varepsilon}, R, \varkappa$, and also on the chosen vector $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and on the vector function $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$.

We introduce the notation $\rho = \tilde{\rho} + \frac{2R}{\varkappa}$, $\rho' = \tilde{\rho} + \frac{4R}{\varkappa}$. Let

$$\Omega_\lambda = \{\tilde{e} \in S_{n-2}(e) : |\tilde{e} - \tilde{e}'| < \frac{2R}{\varkappa} \text{ for some } \tilde{e}' \in \tilde{\Omega}_\lambda\},$$

$$\Omega'_\lambda = \{\tilde{e} \in S_{n-2}(e) : |\tilde{e} - \tilde{e}'| < \frac{4R}{\varkappa} \text{ for some } \tilde{e}' \in \tilde{\Omega}_\lambda\};$$

$\tilde{\Omega}_\lambda \subset \Omega_\lambda \subset \Omega'_\lambda$. The sets Ω'_λ do not intersect for different $\lambda \in \mathcal{L}$. Moreover, $|\tilde{e}' - \tilde{e}''| > \frac{4R}{\varkappa}$ for all $\tilde{e}' \in \Omega_{\lambda_1}$, $\tilde{e}'' \in \Omega_{\lambda_2}$, $\lambda_1 \neq \lambda_2$.

We write

$$\tilde{\mathcal{K}}_\lambda = \tilde{\mathcal{K}}_\lambda(k, \varkappa; \varphi) = \{N \in \mathcal{C}(\frac{1}{2}) : \tilde{e}(k + 2\pi N) \in \tilde{\Omega}_\lambda\},$$

$$\mathcal{K}_\lambda = \mathcal{K}_\lambda(k, \varkappa; \varphi) = \{N \in \mathcal{C}(\frac{1}{2}) : \tilde{e}(k + 2\pi N) \in \Omega_\lambda\},$$

$$\mathcal{K}'_\lambda = \mathcal{K}'_\lambda(k, \varkappa; \varphi) = \{N \in \mathcal{C}(\frac{1}{2}) : \tilde{e}(k + 2\pi N) \in \Omega'_\lambda\}.$$

Property 3b implies that for the vector function $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$, we have (for each sign)

$$\|\hat{G}_\pm^1 \hat{P}^\pm \varphi^{C(\frac{1}{2}) \setminus \bigcup_{\lambda} \tilde{\mathcal{K}}_\lambda}\|^2 \leq \tilde{\varepsilon} \|\hat{G}_\pm^1 \hat{P}^\pm \varphi\|^2. \quad (3.3)$$

Without loss of generality we assume that $\mathcal{E}_2 = e$. For each index $\lambda \in \mathcal{L}$ (and for already chosen k, \varkappa , and φ), we take an orthogonal system of vectors $\mathcal{E}_j^{(\lambda)} \in S_{d-1} \subset \mathbb{R}^d$, $j = 1, \dots, d$, such that $\mathcal{E}_1^{(\lambda)} = \tilde{e}^\lambda$, $\mathcal{E}_2^{(\lambda)} = \mathcal{E}_2 = e$. We let $x_j^{(\lambda)} = (x, \mathcal{E}_j^{(\lambda)})$ denote the coordinates of the vectors $x = \sum_{j=1}^d x_j \mathcal{E}_j \in \mathbb{R}^d$ ($k_j^{(\lambda)}$, $N_j^{(\lambda)}$, $A_j^{(\lambda)}$, and $\tilde{A}_j^{(\lambda)}$ are the coordinates of the vectors $k \in \mathbb{R}^d$, $N \in \Lambda^*$, A , and $\tilde{A}(\tilde{e}^\lambda; \cdot)$). Let $\mathcal{E}_j^{(\lambda)} = \sum_{l=1}^d T_{lj}^{(\lambda)} \mathcal{E}_l$, $j = 1, \dots, d$. Then $A_j^{(\lambda)} = \sum_{l=1}^d T_{lj}^{(\lambda)} A_l$ and $\tilde{A}_j^{(\lambda)}(\cdot) = \sum_{l=1}^d T_{lj}^{(\lambda)} \tilde{A}_l(\tilde{e}^\lambda; \cdot)$. We introduce

the notation $\hat{\alpha}_j^{(\lambda)} = \sum_{l=1}^d T_{lj}^{(\lambda)} \hat{\alpha}_l$, $j = 1, \dots, d$ (the matrices $\hat{\alpha}_j^{(\lambda)} \in \mathcal{S}_M$ satisfy the same commutation relations as the matrices $\hat{\alpha}_j$). The following equality is valid:

$$\widehat{\mathcal{D}}(k + i\kappa e) - \sum_{j=1}^d A_j \hat{\alpha}_j = \sum_{j=1}^d (k_j^{(\lambda)} + i\kappa e_j^{(\lambda)} - i \frac{\partial}{\partial x_j^{(\lambda)}} - A_j^{(\lambda)}) \hat{\alpha}_j^{(\lambda)},$$

where $e_j^{(\lambda)} = 1$ for $j = 2$ and $e_j^{(\lambda)} = 0$ for $j \neq 2$.

For the Fourier coefficients $(\tilde{A}_j^{(\lambda)})_N$ of the functions $\tilde{A}_j^{(\lambda)}$, $j = 1, \dots, d$, we have $(\tilde{A}_j^{(\lambda)})_N = \widehat{\mu}(2\pi N_1^{(\lambda)})(A_j^{(\lambda)})_N$ if $N_2^{(\lambda)} = N_2 = 0$ and $(\tilde{A}_j^{(\lambda)})_N = 0$ if $N_2^{(\lambda)} \neq 0$. (Here, $(A_j^{(\lambda)})_N$ are the Fourier coefficients of $A_j^{(\lambda)}$, $N \in \Lambda^*$).

For $s = 1, 2$ and $\lambda \in \mathcal{L}$, let $\Phi^{(s, \lambda)} : \mathbb{R}^d \rightarrow \mathbb{R}$ be the Λ -periodic trigonometric polynomials with the Fourier coefficients $\Phi_N^{(1, \lambda)} = \Phi_N^{(2, \lambda)} = 0$ if $N_1^{(\lambda)} = N_2^{(\lambda)} = 0$ and

$$\begin{aligned} \Phi_N^{(1, \lambda)} &= (2\pi i ((N_1^{(\lambda)})^2 + (N_2^{(\lambda)})^2))^{-1} (N_1^{(\lambda)}(A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)})_N + N_2^{(\lambda)}(A_2^{(\lambda)} - \tilde{A}_2^{(\lambda)})_N), \\ \Phi_N^{(2, \lambda)} &= -(2\pi i ((N_1^{(\lambda)})^2 + (N_2^{(\lambda)})^2))^{-1} (N_2^{(\lambda)}(A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)})_N - N_1^{(\lambda)}(A_2^{(\lambda)} - \tilde{A}_2^{(\lambda)})_N) \end{aligned}$$

otherwise.

Lemma 3.1. *There is a constant $C^*(\mathfrak{h}) > 0$ such that*

$$\|\Phi^{(s, \lambda)}\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{4} \|\mu\| C^*(\mathfrak{h}), \quad s = 1, 2, \quad \lambda \in \mathcal{L}.$$

Proof. Let $\eta(\cdot) \in C^\infty(\mathbb{R}; \mathbb{R})$, $\eta(\xi) = 0$ for $\xi \leq \pi$, $0 \leq \eta(\xi) \leq 1$ for $\pi < \xi \leq 2\pi$, and $\eta(\xi) = 1$ for $\xi > 2\pi$. For $\xi_1, \xi_2 \in \mathbb{R}$ (and $\xi_1^2 + \xi_2^2 > 0$), we set

$$G(\xi_1, \xi_2) = \frac{\xi_1}{\xi_1^2 + \xi_2^2} \int_0^{+\infty} \frac{\partial \eta(\xi)}{\partial \xi} J_0(\xi \sqrt{\xi_1^2 + \xi_2^2}) d\xi,$$

where $J_0(\cdot)$ is the Bessel function of the first kind of order zero. The estimate

$$|G(\xi_1, \xi_2)| \leq \frac{1}{\sqrt{\xi_1^2 + \xi_2^2}} \quad (3.4)$$

holds. The choice of the function $\eta(\cdot)$ implies that

$$|G(\xi_1, \xi_2)| \cdot (\xi_1^2 + \xi_2^2)^\beta \rightarrow 0 \quad (3.5)$$

as $\xi_1^2 + \xi_2^2 \rightarrow +\infty$ for all $\beta \geq 0$ (whence $G(\cdot, \cdot) \in L^q(\mathbb{R}^2)$, $q \in [1, 2)$). We write

$$G_1(t; \xi_1, \xi_2) = t^{-1} G(t^{-1} \xi_1, t^{-1} \xi_2), \quad t > 0,$$

and $G_2(t; \xi_1, \xi_2) \doteq G_1(t; \xi_2, \xi_1)$. For an arbitrary continuous Λ -periodic function $F : \mathbb{R}^d \rightarrow \mathbb{R}$, we set

$$(F *_\lambda G_s(t; \cdot, \cdot))(x) = \iint_{\mathbb{R}^2} G_s(t; \xi_1, \xi_2) F(x - \xi_1 \tilde{e}^\lambda - \xi_2 e) d\xi_1 d\xi_2, \quad x \in \mathbb{R}^d, \quad s = 1, 2.$$

In this case, $(F *_{\lambda} G_s(t; \cdot, \cdot))_N = 0$ if $N_1^{(\lambda)} = N_2^{(\lambda)} = 0$ and

$$(F *_{\lambda} G_s(t; \cdot, \cdot))_N = - \frac{i N_s^{(\lambda)}}{(N_1^{(\lambda)})^2 + (N_2^{(\lambda)})^2} \eta \left(2\pi t \sqrt{(N_1^{(\lambda)})^2 + (N_2^{(\lambda)})^2} \right) F_N$$

otherwise (here $N \in \Lambda^*$). If $N_2^{(\lambda)} = 0$ and $|N_1^{(\lambda)}| \leq \mathfrak{h}$, then $(A(\cdot) - \tilde{A}(\tilde{e}^{\lambda}; \cdot))_N = 0$. On the other hand, if $N_2^{(\lambda)} \neq 0$, then $|N_2^{(\lambda)}| = |\gamma|^{-1} |(N, \gamma)| \geq |\gamma|^{-1} \geq \mathfrak{h}$. Therefore,

$$2\pi \Phi^{(1, \lambda)} = (A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)}) *_{\lambda} G_1(\mathfrak{h}^{-1}; \cdot, \cdot) + (A_2^{(\lambda)} - \tilde{A}_2^{(\lambda)}) *_{\lambda} G_2(\mathfrak{h}^{-1}; \cdot, \cdot), \quad (3.6)$$

$$2\pi \Phi^{(2, \lambda)} = - (A_1^{(\lambda)} - \tilde{A}_1^{(\lambda)}) *_{\lambda} G_2(\mathfrak{h}^{-1}; \cdot, \cdot) + (A_2^{(\lambda)} - \tilde{A}_2^{(\lambda)}) *_{\lambda} G_1(\mathfrak{h}^{-1}; \cdot, \cdot). \quad (3.7)$$

Estimate (3.4) yields

$$|G_s(\mathfrak{h}^{-1}; \xi_1, \xi_2)| \leq \frac{1}{\sqrt{\xi_1^2 + \xi_2^2}}, \quad s = 1, 2.$$

Hence, from (0.12), for all $x \in \mathbb{R}^d$, we obtain

$$\iint_{\xi_1^2 + \xi_2^2 \leq 1} |G_s(\mathfrak{h}^{-1}; \xi_1, \xi_2)| \cdot |A(x - \xi_1 \tilde{e}^{\lambda} - \xi_2 e)| d\xi_1 d\xi_2 \leq C^*.$$

The last inequality and (3.5) imply that there exists a constant $C^*(\mathfrak{h}) > 0$ (dependent on \mathfrak{h} and the constant C^*) such that for all $x \in \mathbb{R}^d$ (and all $s \in \{1, 2\}$, $\lambda \in \mathcal{L}$), we have

$$\iint_{\mathbb{R}^2} |G_s(\mathfrak{h}^{-1}; \xi_1, \xi_2)| \cdot |A(x - \xi_1 \tilde{e}^{\lambda} - \xi_2 e)| d\xi_1 d\xi_2 \leq \frac{\pi}{8} C^*(\mathfrak{h}).$$

Consequently, we also have

$$\iint_{\mathbb{R}^2} |G_s(\mathfrak{h}^{-1}; \xi_1, \xi_2)| \cdot |\tilde{A}(\tilde{e}^{\lambda}; x - \xi_1 \tilde{e}^{\lambda} - \xi_2 e)| d\xi_1 d\xi_2 \leq \frac{\pi}{8} C^*(\mathfrak{h}).$$

Finally, using (3.6), (3.7), and the inequality $\|\mu\| \geq \hat{\mu}(0) = 1$, for $s = 1, 2$ and $\lambda \in \mathcal{L}$, we derive the claimed estimate. \square

Let us denote

$$\widehat{\mathcal{D}}_0^{(\lambda)} = \left(k_1^{(\lambda)} - i \frac{\partial}{\partial x_1^{(\lambda)}} \right) \widehat{\alpha}_1^{(\lambda)} + \left(k_2^{(\lambda)} + i\kappa - i \frac{\partial}{\partial x_2^{(\lambda)}} \right) \widehat{\alpha}_2^{(\lambda)},$$

$$\widehat{\mathcal{D}}^{(\lambda)} = \widehat{\mathcal{D}}_0^{(\lambda)} - A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} - A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)},$$

$$\widehat{\mathcal{D}}_{\perp}^{(\lambda)} = \sum_{j=3}^n \left(k_j^{(\lambda)} - i \frac{\partial}{\partial x_j^{(\lambda)}} - A_j^{(\lambda)} \right) \widehat{\alpha}_j^{(\lambda)} + \widehat{V}.$$

We have

$$\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W} = \widehat{\mathcal{D}}^{(\lambda)} + \widehat{\mathcal{D}}_{\perp}^{(\lambda)}.$$

We also introduce the notation

$$\widehat{\mathcal{D}}_*^{(\lambda)} = \widehat{\mathcal{D}}_0^{(\lambda)} - \widetilde{A}_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} - \widetilde{A}_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}.$$

Since

$$\frac{\partial \Phi^{(1,\lambda)}}{\partial x_1^{(\lambda)}} - \frac{\partial \Phi^{(2,\lambda)}}{\partial x_2^{(\lambda)}} = A_1^{(\lambda)} - \widetilde{A}_1^{(\lambda)}, \quad \frac{\partial \Phi^{(1,\lambda)}}{\partial x_2^{(\lambda)}} + \frac{\partial \Phi^{(2,\lambda)}}{\partial x_1^{(\lambda)}} = A_2^{(\lambda)} - \widetilde{A}_2^{(\lambda)},$$

the following identity is true:

$$\widehat{\mathcal{D}}^{(\lambda)} = e^{-i \widehat{\alpha}_1^{(\lambda)} \widehat{\alpha}_2^{(\lambda)} \Phi^{(2,\lambda)}} e^{i \Phi^{(1,\lambda)}} \widehat{\mathcal{D}}_*^{(\lambda)} e^{-i \Phi^{(1,\lambda)}} e^{-i \widehat{\alpha}_1^{(\lambda)} \widehat{\alpha}_2^{(\lambda)} \Phi^{(2,\lambda)}}. \quad (3.8)$$

We shall use the brief notation

$$\widehat{P}_\lambda^\pm \doteq \widehat{P}_{\widetilde{e}^\lambda}^\pm = \frac{1}{2} (\widehat{I} \pm i \widehat{\alpha}_1^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}).$$

The relations

$$\widehat{\mathcal{D}}_\perp^{(\lambda)} \widehat{P}_\lambda^\pm = \widehat{P}_\lambda^\pm \widehat{\mathcal{D}}_\perp^{(\lambda)}, \quad \widehat{\mathcal{D}}^{(\lambda)} \widehat{P}_\lambda^\pm = \widehat{P}_\lambda^\mp \widehat{\mathcal{D}}^{(\lambda)}$$

hold (these relations are important for the sequel). For all vector functions $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$, we get

$$\widehat{\mathcal{D}}_0^{(\lambda)} \widehat{P}_\lambda^\pm \psi = \sum_{N \in \Lambda^*} ((k_2^{(\lambda)} + 2\pi N_2^{(\lambda)}) + i(\varkappa \pm (k_1^{(\lambda)} + 2\pi N_1^{(\lambda)}))) \widehat{\alpha}_2^{(\lambda)} \widehat{P}_\lambda^\pm \psi_N e^{2\pi i(N,x)}. \quad (3.9)$$

Let us introduce the operators (acting on the space $L^2(K; \mathbb{C}^M)$)

$$\widehat{G}_{\lambda,\pm}^1 \psi = \sum_{N \in \Lambda^*} ((k_2^{(\lambda)} + 2\pi N_2^{(\lambda)})^2 + (\varkappa \pm |k_1^{(\lambda)} + 2\pi N_1^{(\lambda)}|)^2)^{1/2} \psi_N e^{2\pi i(N,x)}, \quad (3.10)$$

$\psi \in D(\widehat{G}_{\lambda,\pm}^1) = \widetilde{H}^1(K; \mathbb{C}^M) \subset L^2(K; \mathbb{C}^M)$, $\lambda \in \mathcal{L}$. Since $\varkappa \geq \varkappa_0 > (\widetilde{c} + 4)R$, we see that for all $N \in \mathcal{K}'_\lambda$ the condition

$$|\widetilde{e}(k + 2\pi N) - \widetilde{e}^\lambda| < \rho' = \frac{(\widetilde{c} + 4)R}{\varkappa} < 1 \quad (3.11)$$

is fulfilled. Hence, $k_1^{(\lambda)} + 2\pi N_1^{(\lambda)} > 0$ and from (3.9), (3.10), for all vector functions $\psi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{K}'_\lambda)$, we derive

$$\|\widehat{\mathcal{D}}_0^{(\lambda)} \widehat{P}_\lambda^\pm \psi\| = \|\widehat{G}_{\lambda,\pm}^1 \widehat{P}_\lambda^\pm \psi\|. \quad (3.12)$$

Denote

$$b_1 = \frac{1}{2} (\widetilde{c} + 2)R + \frac{3}{4} \frac{|\gamma|}{\pi} (\widetilde{c} + 2)^2 R^2, \quad b_2 = \frac{1}{2} (\widetilde{c} + 4)R + \frac{3}{4} \frac{|\gamma|}{\pi} (\widetilde{c} + 4)^2 R^2.$$

Lemma 3.2. *Given a number $\varkappa \geq \varkappa_0$, a vector $k \in \mathbb{R}^d$, and a vector function $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$, the estimates*

$$\|(\hat{G}_\pm^1 \hat{P}^\pm - \hat{G}_{\lambda, \pm}^1 \hat{P}_\lambda^\pm) \psi\| \leq \frac{b_1}{\varkappa} \|\hat{G}_\pm^1 \psi\|, \quad \psi \in \mathcal{H}(\mathcal{K}_\lambda), \quad (3.13)$$

$$\|(\hat{G}_\pm^1 \hat{P}^\pm - \hat{G}_{\lambda, \pm}^1 \hat{P}_\lambda^\pm) \psi'\| \leq \frac{b_2}{\varkappa} \|\hat{G}_\pm^1 \psi'\|, \quad \psi' \in \mathcal{H}(\mathcal{K}'_\lambda), \quad (3.14)$$

hold for all $\lambda \in \mathcal{L}$.

Proof. For vectors $N \in \mathcal{K}_\lambda$, we have

$$|\tilde{e}(k + 2\pi N) - \tilde{e}^\lambda| < \rho = \frac{(\tilde{c} + 2)R}{\varkappa} < 1.$$

Hence, from (1.4), we obtain

$$\|(\hat{P}^\pm - \hat{P}_\lambda^\pm) \psi\| \leq \rho \|\psi\|, \quad \psi \in \mathcal{H}(\mathcal{K}_\lambda). \quad (3.15)$$

The estimate

$$\|(\hat{P}^\pm - \hat{P}_\lambda^\pm) \psi'\| \leq \rho' \|\psi'\|, \quad \psi' \in \mathcal{H}(\mathcal{K}'_\lambda) \quad (3.16)$$

follows from (3.11) and (1.4).

If $\tilde{e} \in \Omega_\lambda$, then $|\tilde{e} - \tilde{e}^\lambda| < \rho$ and $1 - (\tilde{e}, \tilde{e}^\lambda) = \frac{1}{2} |\tilde{e} - \tilde{e}^\lambda| < \frac{1}{2} \rho^2$. Therefore, for all vectors $N \in \mathcal{K}_\lambda$ (for which $\tilde{e}(k + 2\pi N) \in \Omega_\lambda$), we get

$$\begin{aligned} & | |\varkappa \pm |k_\perp + 2\pi N_\perp| | - |\varkappa \pm (k_1 + 2\pi N_1)| | \leq |k_\perp + 2\pi N_\perp| - (k_1 + 2\pi N_1) = \\ & = (1 - (\tilde{e}(k + 2\pi N), \tilde{e}^\lambda)) |k_\perp + 2\pi N_\perp| < \frac{1}{2} \rho^2 |k_\perp + 2\pi N_\perp| < \frac{3}{4} \rho^2 \varkappa. \end{aligned}$$

Whence

$$\|(\hat{G}_\pm^1 - \hat{G}_{\lambda, \pm}^1) \psi\| \leq \frac{3}{4} \rho^2 \varkappa \|\psi\| = \frac{3}{4} (\tilde{c} + 2)^2 \frac{R^2}{\varkappa} \|\psi\|, \quad \psi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{K}_\lambda). \quad (3.17)$$

Analogously, it follows that

$$\|(\hat{G}_\pm^1 - \hat{G}_{\lambda, \pm}^1) \psi'\| \leq \frac{3}{4} (\tilde{c} + 4)^2 \frac{R^2}{\varkappa} \|\psi'\|, \quad \psi' \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{K}'_\lambda). \quad (3.18)$$

Now, from (3.15) and (3.17), for the vector function $\psi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{K}_\lambda)$, we obtain the inequality (3.13):

$$\begin{aligned} \|(\hat{G}_\pm^1 \hat{P}^\pm - \hat{G}_{\lambda, \pm}^1 \hat{P}_\lambda^\pm) \psi\| & \leq \|(\hat{P}^\pm - \hat{P}_\lambda^\pm) \hat{G}_\pm^1 \psi\| + \|(\hat{G}_\pm^1 - \hat{G}_{\lambda, \pm}^1) \hat{P}_\lambda^\pm \psi\| \leq \\ & \leq \frac{1}{2} \rho \|\hat{G}_\pm^1 \psi\| + \frac{3}{4} \rho^2 \varkappa \|\hat{P}_\lambda^\pm \psi\| \leq \\ & \leq \frac{1}{2} \frac{(\tilde{c} + 2)R}{\varkappa} \|\hat{G}_\pm^1 \psi\| + \frac{3}{4} \frac{(\tilde{c} + 2)^2 R^2}{\varkappa} \|\psi\| \leq \frac{b_1}{\varkappa} \|\hat{G}_\pm^1 \psi\|. \end{aligned}$$

Inequality (3.14) is proved similarly (using estimates (3.16) and (3.18)). \square

In the sequel, we shall use the notation

$$\widehat{P}_\lambda^\pm \widehat{P}^{\kappa_\lambda} \varphi = \widehat{P}_\lambda^\pm \varphi^{\kappa_\lambda} = \varphi_\lambda^\pm, \quad \widehat{P}_\lambda^\pm \widehat{P}^{\tilde{\kappa}_\lambda} \varphi = \widehat{P}_\lambda^\pm \varphi^{\tilde{\kappa}_\lambda} = \tilde{\varphi}_\lambda^\pm.$$

On the space $L^2(K; \mathbb{C}^M)$, we define the orthogonal projections $\widehat{P}^{\{\Omega_\lambda\}}$, $\lambda \in \mathcal{L}$, that take a vector function $\psi \in L^2(K; \mathbb{C}^M)$ to the vector function $\widehat{P}^{\{\Omega_\lambda\}} \psi$ for which $(\widehat{P}^{\{\Omega_\lambda\}} \psi)_N = \psi_N$ if $\tilde{e}(k + 2\pi N) \in \Omega_\lambda$ and $(\widehat{P}^{\{\Omega_\lambda\}} \psi)_N = 0$ if either the vector $\tilde{e}(k + 2\pi N)$ is not defined (for $k_\perp + 2\pi N_\perp = 0$) or $\tilde{e}(k + 2\pi N) \notin \Omega_\lambda$. By analogy with the orthogonal projections $\widehat{P}^{\{\Omega_\lambda\}}$, define the orthogonal projections $\widehat{P}^{\{\tilde{\Omega}_\lambda\}}$ (replacing the sets $\Omega_\lambda \subset S_{n-2}(e)$ by the sets $\tilde{\Omega}_\lambda$).

For all vector functions $\psi \in L^2(K; \mathbb{C}^M)$, we have

$$\|(\widehat{P}^\pm - \widehat{P}_\lambda^\pm) \widehat{P}^{\{\Omega_\lambda\}} \psi\| \leq \frac{1}{2} \rho \|\widehat{P}^{\{\Omega_\lambda\}} \psi\|. \quad (3.19)$$

Since $\varkappa \geq \varkappa_0 > 4R$ (and $\varphi \in \tilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$), we see that in the case, where $(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})_N \neq 0$, $N \in \Lambda^*$, the estimate

$$|k_\perp + 2\pi N_\perp| > \frac{\varkappa}{2} - R > \frac{\varkappa}{4}$$

holds.

Lemma 3.3. *If $N \in \mathcal{K}_\lambda$, then*

$$\left| \sum_{j=3}^d (k_j^{(\lambda)} + 2\pi N_j^{(\lambda)}) \mathcal{E}_j^{(\lambda)} \right| < \frac{3}{2} (\tilde{c} + 2)R. \quad (3.20)$$

Proof. Indeed,

$$\left| \sum_{j=3}^d (k_j^{(\lambda)} + 2\pi N_j^{(\lambda)}) \mathcal{E}_j^{(\lambda)} \right| = |k_\perp + 2\pi N_\perp - (k_\perp + 2\pi N_\perp, \tilde{e}^\lambda) \tilde{e}^\lambda| \leq \quad (3.21)$$

$$\leq |\tilde{e}(k + 2\pi N) - \tilde{e}^\lambda| \cdot |k_\perp + 2\pi N_\perp| < \rho |k_\perp + 2\pi N_\perp| = \frac{(\tilde{c} + 2)R}{\varkappa} |k_\perp + 2\pi N_\perp|.$$

At the same time, the definition of the set $\mathcal{C}(\frac{1}{2})$ implies that $|k_\perp + 2\pi N_\perp| < \frac{3}{2} \varkappa$. Therefore inequality (3.20) follows from (3.21). \square

From estimate (1.2) (under the change $k \rightarrow k - \varkappa \tilde{e}^\lambda$) and Lemma 3.3, for all $\varepsilon > 0$, we obtain

$$\begin{aligned} \|\widehat{W} \varphi_\lambda^\pm\| &\leq \|\widehat{V} \varphi_\lambda^\pm\| + \|A\| \varphi_\lambda^\pm \leq \\ &\leq \varepsilon \left\| \sum_{j=1}^d (k_j - \varkappa \tilde{e}_j^\lambda - i \frac{\partial}{\partial x_j}) \widehat{\alpha}_j \varphi_\lambda^\pm \right\| + C'_\varepsilon(0, \widehat{W}) \|\varphi_\lambda^\pm\| \leq \\ &\leq \varepsilon \left\| ((k_1^{(\lambda)} - \varkappa - i \frac{\partial}{\partial x_1^{(\lambda)}}) \widehat{\alpha}_1^{(\lambda)} + (k_2^{(\lambda)} - i \frac{\partial}{\partial x_2^{(\lambda)}}) \widehat{\alpha}_2^{(\lambda)}) \varphi_\lambda^\pm \right\| + \end{aligned} \quad (3.22)$$

$$\begin{aligned}
& + \varepsilon \left\| \sum_{j=3}^d (k_j^{(\lambda)} - i \frac{\partial}{\partial x_j^{(\lambda)}}) \widehat{\alpha}_j^{(\lambda)} \varphi_\lambda^\pm \right\| + C'_\varepsilon(0, \widehat{W}) \|\varphi_\lambda^\pm\| \leq \\
& \leq \varepsilon \|\widehat{G}_{\lambda,-}^1 \varphi_\lambda^\pm\| + \left(\frac{3\varepsilon}{2} (\widetilde{c} + 2)R + C'_\varepsilon(0, \widehat{W}) \right) \|\varphi_\lambda^\pm\|.
\end{aligned}$$

Whence

$$\begin{aligned}
\|\widehat{\mathcal{D}}_\perp^{(\lambda)} \varphi_\lambda^\pm\| & \leq \|\widehat{W} \varphi_\lambda^\pm\| + \left\| \sum_{j=3}^d (k_j^{(\lambda)} - i \frac{\partial}{\partial x_j^{(\lambda)}}) \widehat{\alpha}_j^{(\lambda)} \varphi_\lambda^\pm \right\| \leq \\
& \leq \varepsilon \|\widehat{G}_{\lambda,-}^1 \varphi_\lambda^\pm\| + \left(\frac{3(\varepsilon+1)}{2} (\widetilde{c} + 2)R + C'_\varepsilon(0, \widehat{W}) \right) \|\varphi_\lambda^\pm\|.
\end{aligned} \tag{3.23}$$

Given $N \in \mathcal{K}'_\lambda$, the following inequality is proved in the same way as inequality (3.20) (see the proof of Lemma 3.3):

$$\left| \sum_{j=3}^d (k_j^{(\lambda)} + 2\pi N_j^{(\lambda)}) \mathcal{E}_j^{(\lambda)} \right| < \frac{3}{2} (\widetilde{c} + 4)R.$$

Therefore, by analogy with inequality (3.23), for all $\varepsilon > 0$ and all vector functions $\psi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{K}'_\lambda)$, we also derive

$$\|\widehat{\mathcal{D}}_\perp^{(\lambda)} \psi\| \leq \varepsilon \|\widehat{G}_{\lambda,-}^1 \psi\| + C_3^\sharp(\varepsilon) \|\psi\|, \tag{3.24}$$

where $C_3^\sharp(\varepsilon) = \frac{3(\varepsilon+1)}{2} (\widetilde{c} + 4)R + C'_\varepsilon(0, \widehat{W})$.

We write

$$C'_3 = 3(\widetilde{c} + 2)R + C'_1(0, \widehat{W}), \quad C_4 = 1 + \frac{|\gamma|}{\pi} C'_3.$$

Choose a number $a \in (0, 1)$ such that

$$a^2 \max \left\{ C_4^2, \frac{9}{4} \frac{|\gamma|^2}{\pi^2} b_2^2 \right\} < \frac{\delta \delta_1}{50} (1 - \widetilde{\varepsilon}) C_2^2.$$

Suppose a number $\varepsilon_1 \in (0, \frac{1}{2}]$ satisfies the inequalities

$$(2\varepsilon_1)^2 < \frac{\delta \delta_1}{40} (1 - \widetilde{\varepsilon}) a^2, \quad \frac{9}{2} \frac{|\gamma|^2}{\pi^2} b_2^2 (2\varepsilon_1)^2 < \frac{\delta \delta_1}{50} (1 - \widetilde{\varepsilon}) C_2^2.$$

Let us denote

$$C'_3(\varepsilon_1) = \frac{3\varepsilon_1}{2} (\widetilde{c} + 2)R + C'_{\varepsilon_1}(0, \widehat{W}).$$

From (3.16), (3.22) (for $\varepsilon = \varepsilon_1$), and (3.23) (for $\varepsilon = 1$), it follows that

$$\begin{aligned}
& \|\widehat{P}^{\{\widetilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa_\lambda}\| = \\
& = \|\widehat{P}^{\{\widetilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}^{(\lambda)} + \widehat{\mathcal{D}}_\perp^{(\lambda)}) \varphi^{\kappa_\lambda}\| \geq \|\widehat{P}^{\{\widetilde{\Omega}_\lambda\}} \widehat{\mathcal{D}}^{(\lambda)} \varphi_\lambda^+\| - \|\widehat{P}^{\{\widetilde{\Omega}_\lambda\}} \widehat{\mathcal{D}}_\perp^{(\lambda)} \varphi_\lambda^-\| \geq \\
& \geq \|\widehat{P}^{\{\widetilde{\Omega}_\lambda\}} \widehat{\mathcal{D}}_0^{(\lambda)} \varphi_\lambda^+\| - \|(A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \varphi_\lambda^+\| - \|\widehat{\mathcal{D}}_\perp^{(\lambda)} \varphi_\lambda^-\| \geq \\
& \geq \|\widehat{\mathcal{D}}_0^{(\lambda)} \widetilde{\varphi}_\lambda^+\| - \| |A| \varphi_\lambda^+\| - \|\widehat{\mathcal{D}}_\perp^{(\lambda)} \varphi_\lambda^-\| \geq \\
& \geq \|\widehat{G}_{\lambda,+}^1 \widetilde{\varphi}_\lambda^+\| - (\varepsilon_1 \|\widehat{G}_{\lambda,-}^1 \varphi_\lambda^+\| + C'_3(\varepsilon_1) \|\varphi_\lambda^+\|) - (\|\widehat{G}_{\lambda,-}^1 \varphi_\lambda^-\| + C'_3 \|\varphi_\lambda^-\|).
\end{aligned} \tag{3.25}$$

Lemma 3.4. For all $\lambda \in \mathcal{L}$,

$$\|\widehat{\mathcal{D}}^{(\lambda)} \widetilde{\varphi}_\lambda^-\| \geq C_2 \|\widehat{G}_{\lambda,-}^1 \widetilde{\varphi}_\lambda^-\|.$$

Proof. From (3.12) it follows that

$$\|\widehat{\mathcal{D}}_0^{(\lambda)} \widetilde{\varphi}_\lambda^-\| = \|\widehat{G}_{\lambda,-}^1 \widetilde{\varphi}_\lambda^-\|. \quad (3.26)$$

By (1.9), (3.9), and the equality (3.26), we obtain

$$\begin{aligned} & \|(A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \widetilde{\varphi}_\lambda^-\| \leq \| |A| \widetilde{\varphi}_\lambda^-\| \leq \\ & \leq \tau \left(\left\| \left(k_1^{(\lambda)} - \varkappa - i \frac{\partial}{\partial x_1^{(\lambda)}} \right) \widetilde{\varphi}_\lambda^- \right\|^2 + \left\| \left(k_2^{(\lambda)} - i \frac{\partial}{\partial x_2^{(\lambda)}} \right) \widetilde{\varphi}_\lambda^- \right\|^2 \right)^{1/2} + Q \|\widetilde{\varphi}_\lambda^-\| \leq \\ & \leq \tau \|\widehat{\mathcal{D}}_0^{(\lambda)} \widetilde{\varphi}_\lambda^-\| + Q \|\widetilde{\varphi}_\lambda^-\| = \tau \|\widehat{G}_{\lambda,-}^1 \widetilde{\varphi}_\lambda^-\| + Q \|\widetilde{\varphi}_\lambda^-\|. \end{aligned} \quad (3.27)$$

Whence

$$\begin{aligned} \|\widehat{\mathcal{D}}^{(\lambda)} \widetilde{\varphi}_\lambda^-\| & \geq \|\widehat{\mathcal{D}}_0^{(\lambda)} \widetilde{\varphi}_\lambda^-\| - \|(A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \widetilde{\varphi}_\lambda^-\| \geq \\ & \geq (1 - \tau) \|\widehat{G}_{\lambda,-}^1 \widetilde{\varphi}_\lambda^-\| - Q \|\widetilde{\varphi}_\lambda^-\|. \end{aligned} \quad (3.28)$$

Now, we use identity (3.8). Denote

$$\chi_\lambda = e^{-i\Phi^{(1,\lambda)}} e^{-i\widehat{\alpha}_1^{(\lambda)} \widehat{\alpha}_2^{(\lambda)} \Phi^{(2,\lambda)}} \widetilde{\varphi}_\lambda^-.$$

Since the functions $\Phi^{(s,\lambda)}$, $s = 1, 2$, $\lambda \in \mathcal{L}$, are trigonometric polynomials (and $\Phi_N^{(s,\lambda)} = 0$ for all vectors $N \in \Lambda^*$ with $A_N = 0$), we have $\chi_\lambda \in \widetilde{H}^1(K; \mathbb{C}^M)$. Furthermore, the operator \widehat{P}_λ^- commutes with the operators of multiplication by the function $e^{-i\Phi^{(1,\lambda)}}$ and by the matrix function $e^{-i\widehat{\alpha}_1^{(\lambda)} \widehat{\alpha}_2^{(\lambda)} \Phi^{(2,\lambda)}}$. Using Lemma 3.1, inequality (0.13) (condition **(A₂)**), and inequality

$$\|\widehat{\mathcal{D}}_0^{(\lambda)} \chi_\lambda\| \geq \frac{\pi}{|\gamma|} \|\chi_\lambda\|, \quad (3.29)$$

which is a consequence of the choice of the vector $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$ (see (3.9)), we get

$$\begin{aligned} \|\widehat{\mathcal{D}}^{(\lambda)} \widetilde{\varphi}_\lambda^-\| & \geq e^{-\frac{1}{2}\|\mu\|C^*(h)} \|\widehat{\mathcal{D}}_*^{(\lambda)} \chi_\lambda\| \geq \\ & \geq e^{-\frac{1}{2}\|\mu\|C^*(h)} \left(\|\widehat{\mathcal{D}}_0^{(\lambda)} \chi_\lambda\| - \|(\widetilde{A}_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + \widetilde{A}_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \chi_\lambda\| \right) \geq \\ & \geq e^{-\frac{1}{2}\|\mu\|C^*(h)} \left(\|\widehat{\mathcal{D}}_0^{(\lambda)} \chi_\lambda\| - \|\widetilde{A}(\widetilde{e}^\lambda; \cdot) \chi_\lambda(\cdot)\| \right) \geq \\ & \geq e^{-\frac{1}{2}\|\mu\|C^*(h)} (1 - \widetilde{\theta}) \frac{\pi}{|\gamma|} \|\chi_\lambda\| \geq (1 - \widetilde{\theta}) \frac{\pi}{|\gamma|} e^{-\|\mu\|C^*(h)} \|\widetilde{\varphi}_\lambda^-\|. \end{aligned}$$

Multiplying inequality (3.28) by $(1 - \widetilde{\theta}) \frac{\pi}{|\gamma|} e^{-\|\mu\|C^*(h)}$, multiplying inequality (3.29) by Q , and adding them together, we derive the claimed estimate. \square

Now, let us get estimate (3.33), which complements estimate (3.25). By (3.2), (3.12), and Lemma 3.4, we have

$$\begin{aligned}
& \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{\mathcal{D}}^{(\lambda)} \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda} \| = \\
& = \| \widehat{P}^{\{\Omega_\lambda\}} (\widehat{\mathcal{D}}_0^{(\lambda)} - A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} - A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda} \| = \\
& = \| \widehat{\mathcal{D}}_0^{(\lambda)} \widehat{P}_\lambda^- (\varphi^{\mathcal{K}_\lambda \setminus \tilde{\mathcal{K}}_\lambda} + \varphi^{\tilde{\mathcal{K}}_\lambda}) - \widehat{P}^{\{\Omega_\lambda\}} (A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \widehat{P}_\lambda^- (\varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} + \varphi^{\tilde{\mathcal{K}}_\lambda}) \| = \\
& = \| \widehat{\mathcal{D}}^{(\lambda)} \tilde{\varphi}_\lambda^- + \widehat{\mathcal{D}}_0^{(\lambda)} \widehat{P}_\lambda^- \varphi^{\mathcal{K}_\lambda \setminus \tilde{\mathcal{K}}_\lambda} - \widehat{P}^{\{\Omega_\lambda\}} (A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| \geq \\
& \geq \| \widehat{\mathcal{D}}^{(\lambda)} \tilde{\varphi}_\lambda^- \| - \| \widehat{\mathcal{D}}_0^{(\lambda)} \widehat{P}_\lambda^- \varphi^{\mathcal{K}_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| - \| (A_1^{(\lambda)} \widehat{\alpha}_1^{(\lambda)} + A_2^{(\lambda)} \widehat{\alpha}_2^{(\lambda)}) \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| \geq \\
& \geq C_2 \| \widehat{G}_{\lambda,-}^1 - \tilde{\varphi}_\lambda^- \| - \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^- \varphi^{\mathcal{K}_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| - \| |A| \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \|.
\end{aligned} \tag{3.30}$$

At the same time, from (1.9), (3.9) and (3.12) (by analogy with the estimate (3.27)) it follows that

$$\| |A| \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| \leq \tau \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| + Q \| \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \|, \tag{3.31}$$

and (for $\varepsilon = \varepsilon_1$ and $\psi = \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda}$) inequality (3.24) implies

$$\| \widehat{\mathcal{D}}_\perp^{(\lambda)} \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| \leq \varepsilon_1 \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| + C_3^\sharp(\varepsilon_1) \| \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \|. \tag{3.32}$$

Whence (see (3.30), (3.31), (3.32))

$$\begin{aligned}
& \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\mathcal{K}'_\lambda} \| = \\
& = \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}^{(\lambda)} + \widehat{\mathcal{D}}_\perp^{(\lambda)}) \varphi^{\mathcal{K}'_\lambda} \| \geq \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{\mathcal{D}}^{(\lambda)} \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda} \| - \| \widehat{\mathcal{D}}_\perp^{(\lambda)} \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| \geq \\
& \geq C_2 \| \widehat{G}_{\lambda,-}^1 - \tilde{\varphi}_\lambda^- \| - \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^- \varphi^{\mathcal{K}_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| - (\tau \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| + Q \| \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \|) - \\
& \quad - (\varepsilon_1 \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| + C_3^\sharp(\varepsilon_1) \| \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \|) \geq \\
& \geq C_2 \| \widehat{G}_{\lambda,-}^1 - \tilde{\varphi}_\lambda^- \| - C_3 \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^- \varphi^{\mathcal{K}_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \| - \varepsilon_1 \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| - C_3^\sharp(\varepsilon_1) \| \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \|.
\end{aligned} \tag{3.33}$$

In what follows, we assume that $C_3^\sharp(\varepsilon_1) \leq \varepsilon_1 \varkappa_0$. Then $C_3'(\varepsilon_1) \leq \varepsilon_1 \varkappa_0$ as well. Since for all vector functions $\psi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{K}'_\lambda)$ the inequalities $\| \widehat{G}_{\lambda,-}^1 - \psi \| \leq \| \widehat{G}_{\lambda,+}^1 + \psi \|$ and $\varkappa \| \psi \| \leq \| \widehat{G}_{\lambda,+}^1 + \psi \|$ hold, we derive (for all $\varkappa \geq \varkappa_0$ and all $\lambda \in \mathcal{L}$)

$$\begin{aligned}
& \varepsilon_1 \| \widehat{G}_{\lambda,-}^1 - \varphi_\lambda^+ \| + C_3'(\varepsilon_1) \| \varphi_\lambda^+ \| \leq 2\varepsilon_1 \| \widehat{G}_{\lambda,+}^1 + \varphi_\lambda^+ \|, \\
& \varepsilon_1 \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| + C_3^\sharp(\varepsilon_1) \| \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| \leq 2\varepsilon_1 \| \widehat{G}_{\lambda,+}^1 + \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \|.
\end{aligned}$$

Hence, from (3.25) and (3.31) it follows that

$$\begin{aligned}
& \| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\mathcal{K}_\lambda} \| \geq \| \widehat{G}_{\lambda,+}^1 - \tilde{\varphi}_\lambda^+ \| - 2\varepsilon_1 \| \widehat{G}_{\lambda,+}^1 + \varphi_\lambda^+ \| - C_4 \| \widehat{G}_{\lambda,-}^1 - \varphi_\lambda^- \|, \\
& \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\mathcal{K}'_\lambda} \| \geq \\
& \geq C_2 \| \widehat{G}_{\lambda,-}^1 - \tilde{\varphi}_\lambda^- \| - 2\varepsilon_1 \| \widehat{G}_{\lambda,+}^1 - \widehat{P}_\lambda^+ \varphi^{\mathcal{K}'_\lambda} \| - C_3 \| \widehat{G}_{\lambda,-}^1 - \widehat{P}_\lambda^- \varphi^{\mathcal{K}'_\lambda \setminus \tilde{\mathcal{K}}_\lambda} \|.
\end{aligned}$$

Using Lemma 3.2 (inequalities (3.13) and (3.14)), from the last estimate we get

$$\| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa_\lambda} \| \geq \quad (3.34)$$

$$\begin{aligned} &\geq \| \widehat{G}_+^1 \widehat{P}^+ \varphi^{\tilde{\kappa}_\lambda} \| - 2\varepsilon_1 \| \widehat{G}_+^1 \widehat{P}^+ \varphi^{\kappa_\lambda} \| - C_4 \| \widehat{G}_-^1 \widehat{P}^- \varphi^{\kappa_\lambda} \| - \\ &\quad - \frac{b_1}{\varkappa} (\| \widehat{G}_+^1 \varphi^{\tilde{\kappa}_\lambda} \| + 2\varepsilon_1 \| \widehat{G}_+^1 \varphi^{\kappa_\lambda} \| + C_4 \| \widehat{G}_-^1 \varphi^{\kappa_\lambda} \|), \end{aligned}$$

$$\| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa'_\lambda} \| \geq \quad (3.35)$$

$$\begin{aligned} &\geq C_2 \| \widehat{G}_-^1 \widehat{P}^- \varphi^{\tilde{\kappa}_\lambda} \| - 2\varepsilon_1 \| \widehat{G}_+^1 \widehat{P}^+ \varphi^{\kappa'_\lambda} \| - C_3 \| \widehat{G}_-^1 \widehat{P}^- \varphi^{\kappa'_\lambda \setminus \tilde{\kappa}_\lambda} \| - \\ &\quad - \frac{b_2}{\varkappa} (C_2 \| \widehat{G}_-^1 \varphi^{\tilde{\kappa}_\lambda} \| + 2\varepsilon_1 \| \widehat{G}_+^1 \varphi^{\kappa'_\lambda} \| + C_3 \| \widehat{G}_-^1 \varphi^{\kappa'_\lambda \setminus \tilde{\kappa}_\lambda} \|). \end{aligned}$$

From (3.2) and (3.19) (for a number $\varkappa \geq \varkappa_0$, a vector $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and a vector function $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$) we obtain

$$\begin{aligned} &\| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \| \geq \\ &\geq \| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa_\lambda} \| - \frac{\rho}{2} \| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|, \\ &\| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \| \geq \\ &\geq \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa'_\lambda} \| - \frac{\rho}{2} \| \widehat{P}^{\{\Omega_\lambda\}} (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|. \end{aligned}$$

Whence

$$\begin{aligned} &\| \widehat{P}^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 + a^2 \| \widehat{P}^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 \geq \quad (3.36) \\ &\geq \sum_\lambda \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 + \\ &\quad + a^2 \sum_\lambda \| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 \geq \\ &\geq (1 - \delta_1) \sum_\lambda \| \widehat{P}^{\{\Omega_\lambda\}} \widehat{P}_\lambda^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa'_\lambda} \|^2 + \\ &\quad + (1 - \delta_1) a^2 \sum_\lambda \| \widehat{P}^{\{\tilde{\Omega}_\lambda\}} \widehat{P}_\lambda^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi^{\kappa_\lambda} \|^2 - \\ &\quad - \frac{(1 - \delta_1)}{\delta_1} \frac{1 + a^2}{4} \rho^2 \sum_\lambda \| \widehat{P}^{\{\Omega_\lambda\}} (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\sum_\lambda \| \widehat{P}^{\{\Omega_\lambda\}} (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 \leq \| (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 \leq \quad (3.37) \\ &\leq \frac{1}{a^2} (\| \widehat{P}^+ (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2 + a^2 \| \widehat{P}^- (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}) \varphi \|^2). \end{aligned}$$

Next, suppose the number \varkappa_0 satisfies the condition

$$\frac{(1-\delta_1)}{\delta_1} \frac{1+a^2}{4a^2} (\tilde{c}+2)^2 R^2 \leq \frac{\delta_2-\delta_1}{1-\delta_2} \varkappa_0^2$$

and $\varkappa \geq \varkappa_0$, then (3.36) and (3.37) yield

$$\begin{aligned} & (1-\delta_2)^{-1} \left(\|\hat{P}^+(\widehat{\mathcal{D}}(k+i\varkappa e)+\widehat{W})\varphi\|^2 + a^2 \|\hat{P}^-(\widehat{\mathcal{D}}(k+i\varkappa e)+\widehat{W})\varphi\|^2 \right) \geq \quad (3.38) \\ & \geq \sum_{\lambda} \|\hat{P}^{\{\Omega_{\lambda}\}} \hat{P}_{\lambda}^+(\widehat{\mathcal{D}}(k+i\varkappa e)+\widehat{W})\varphi^{\kappa'_{\lambda}}\|^2 + a^2 \sum_{\lambda} \|\hat{P}^{\{\tilde{\Omega}_{\lambda}\}} \hat{P}_{\lambda}^-(\widehat{\mathcal{D}}(k+i\varkappa e)+\widehat{W})\varphi^{\kappa_{\lambda}}\|^2. \end{aligned}$$

On the other hand, from estimates (3.34) and (3.35) it follows that the right hand side of previous inequality (3.38) is greater than or equal to

$$\begin{aligned} & (1-\delta_1) \left(C_2^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi^{\tilde{\kappa}_{\lambda}}\|^2 + a^2 \sum_{\lambda} \|\widehat{G}_{+}^1 \widehat{P}^{+}\varphi^{\tilde{\kappa}_{\lambda}}\|^2 \right) - \quad (3.39) \\ & - \frac{5(1-\delta_1)}{\delta_1} \left((2\varepsilon_1)^2 \sum_{\lambda} \|\widehat{G}_{+}^1 \widehat{P}^{+}\varphi^{\kappa'_{\lambda}}\|^2 + C_3^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi^{\kappa'_{\lambda} \setminus \tilde{\kappa}_{\lambda}}\|^2 + \right. \\ & \quad \left. + a^2 (2\varepsilon_1)^2 \sum_{\lambda} \|\widehat{G}_{+}^1 \widehat{P}^{+}\varphi^{\kappa_{\lambda}}\|^2 + a^2 C_4^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi^{\kappa_{\lambda}}\|^2 + \right. \\ & \quad \left. + \frac{b_2^2}{\varkappa^2} \left(C_2^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \varphi^{\tilde{\kappa}_{\lambda}}\|^2 + (2\varepsilon_1)^2 \sum_{\lambda} \|\widehat{G}_{+}^1 \varphi^{\kappa'_{\lambda}}\|^2 + C_3^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \varphi^{\kappa'_{\lambda} \setminus \tilde{\kappa}_{\lambda}}\|^2 \right) + \right. \\ & \quad \left. + \frac{a^2 b_1^2}{\varkappa^2} \left(\sum_{\lambda} \|\widehat{G}_{+}^1 \varphi^{\tilde{\kappa}_{\lambda}}\|^2 + (2\varepsilon_1)^2 \sum_{\lambda} \|\widehat{G}_{+}^1 \varphi^{\kappa_{\lambda}}\|^2 + C_4^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \varphi^{\kappa_{\lambda}}\|^2 \right) \right) \geq \\ & \geq (1-\delta_1) \left(C_2^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi^{\tilde{\kappa}_{\lambda}}\|^2 + a^2 \sum_{\lambda} \|\widehat{G}_{+}^1 \widehat{P}^{+}\varphi^{\tilde{\kappa}_{\lambda}}\|^2 \right) - \\ & - \frac{5(1-\delta_1)}{\delta_1} \left((1+a^2)(2\varepsilon_1)^2 \|\widehat{G}_{+}^1 \widehat{P}^{+}\varphi\|^2 + C_3^2 \sum_{\lambda} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi^{\kappa'_{\lambda} \setminus \tilde{\kappa}_{\lambda}}\|^2 + \right. \\ & \quad \left. + a^2 C_4^2 \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi\|^2 + \frac{1}{\varkappa^2} (b_2^2 C_2^2 + b_2^2 C_3^2 + a^2 b_1^2 C_4^2) \|\widehat{G}_{-}^1 \varphi\|^2 + \right. \\ & \quad \left. + \frac{1}{\varkappa^2} (b_2^2 (2\varepsilon_1)^2 + a^2 b_1^2 + a^2 b_1^2 (2\varepsilon_1)^2) \|\widehat{G}_{+}^1 \varphi\|^2 \right). \end{aligned}$$

Let us now use condition 3b (also see (3.3)) which yields

$$\begin{aligned} & \sum_{\lambda} \|\widehat{G}_{\pm}^1 \widehat{P}^{\pm}\varphi^{\tilde{\kappa}_{\lambda}}\|^2 \geq (1-\tilde{\varepsilon}) \|\widehat{G}_{\pm}^1 \widehat{P}^{\pm}\varphi\|^2, \\ & \sum_{\lambda} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi^{\kappa'_{\lambda} \setminus \tilde{\kappa}_{\lambda}}\|^2 \leq \tilde{\varepsilon} \|\widehat{G}_{-}^1 \widehat{P}^{-}\varphi\|^2. \end{aligned}$$

Since

$$\begin{aligned}\|\widehat{G}_-^1\varphi\|^2 &= \|\widehat{G}_-^1\widehat{P}^-\varphi\|^2 + \|\widehat{G}_-^1\widehat{P}^+\varphi\|^2 \leq \|\widehat{G}_-^1\widehat{P}^-\varphi\|^2 + \|\widehat{G}_+^1\widehat{P}^-\varphi\|^2, \\ \|\widehat{G}_+^1\varphi\|^2 &= \|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 + \|\widehat{G}_+^1\widehat{P}^-\varphi\|^2 \leq \|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 + \frac{9}{4}\varkappa^2\|\widehat{P}_-^1\varphi\|^2 \leq \\ &\leq \|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 + \frac{9}{4}\frac{|\gamma|^2}{\pi^2}\varkappa^2\|\widehat{G}_-^1\widehat{P}^-\varphi\|^2,\end{aligned}$$

and $C_2 \in (0, 1)$, $a \in (0, 1)$, $\varepsilon_1 \in (0, \frac{1}{2}]$, $b_1 < b_2$, using (3.38) and (3.39) we deduce the estimate

$$\begin{aligned}(1 - \delta_2)^{-1} \left(\|\widehat{P}^+(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 + a^2\|\widehat{P}^-(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 \right) &\geq \quad (3.40) \\ &\geq (1 - \delta_1)(1 - \widetilde{\varepsilon}) \left(C_2^2\|\widehat{G}_-^1\widehat{P}^-\varphi\|^2 + a^2\|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 \right) - \\ &\quad - \frac{5(1 - \delta_1)}{\delta_1} \left((\widetilde{\varepsilon}C_3^2 + a^2C_4^2 + \frac{9}{4}\frac{|\gamma|^2}{\pi^2}(2\varepsilon_1)^2 + a^2)b_2^2 + \right. \\ &\quad \left. + (1 + C_3^2 + C_4^2)\frac{b_2^2}{\varkappa^2} \right) \|\widehat{G}_-^1\widehat{P}^-\varphi\|^2 + (2\varepsilon_1)^2 + (4 + C_3^2 + C_4^2)\frac{b_2^2}{\varkappa^2} \|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 \Big). \end{aligned}$$

Finally, suppose that the number \varkappa_0 also satisfies the conditions

$$(1 + C_3^2 + C_4^2)b_2^2 \leq \frac{\delta_1}{5}(1 - \widetilde{\varepsilon})C_2^2\frac{\delta}{10}\varkappa_0^2, \quad (4 + C_3^2 + C_4^2)b_2^2 \leq \frac{\delta_1}{5}(1 - \widetilde{\varepsilon})a^2\frac{\delta}{10}\varkappa_0^2.$$

From the choice of the numbers $\widetilde{\varepsilon}$, a , and ε_1 it follows that

$$\begin{aligned}2(2\varepsilon_1)^2 &< \frac{\delta_1}{5}(1 - \widetilde{\varepsilon})a^2\frac{\delta}{4}, \\ \max\left\{\widetilde{\varepsilon}C_3^2, a^2C_4^2, \frac{9}{2}\frac{|\gamma|^2}{\pi^2}(2\varepsilon_1)^2b_2^2, \frac{9}{4}\frac{|\gamma|^2}{\pi^2}a^2b_2^2\right\} &< \frac{\delta_1}{5}(1 - \widetilde{\varepsilon})C_2^2\frac{\delta}{10}\end{aligned}$$

and $\widetilde{\varepsilon} \leq \frac{\delta}{8}$. Therefore, from (3.40) for all $\varkappa \geq \varkappa_0$ (all $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M) \cap \mathcal{H}(\mathcal{C}(\frac{1}{2}))$) we get the inequality

$$\begin{aligned}&\|\widehat{P}^+(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 + a^2\|\widehat{P}^-(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W})\varphi\|^2 \geq \\ &\geq (1 - \delta_2)(1 - \delta_1)(1 - \widetilde{\varepsilon})\left(1 - \frac{\delta}{2}\right) \left(C_2^2\|\widehat{G}_-^1\widehat{P}^-\varphi\|^2 + a^2\|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 \right) \geq \\ &\geq (1 - \delta) \left(C_2^2\|\widehat{G}_-^1\widehat{P}^-\varphi\|^2 + a^2\|\widehat{G}_+^1\widehat{P}^+\varphi\|^2 \right).\end{aligned}$$

This completes the proof of Theorem 2.1.

4 Proof of Theorem 1.4

Lemma 4.1. *Let $d \geq 3$. Suppose $\gamma \in \Lambda \setminus \{0\}$, $\widehat{W} \in L^2(K; \mathcal{M}_M)$, and $||| \widehat{W} |||_{\gamma, M} < +\infty$. Then there is a number $c^* = c^*(\gamma) > 0$ such that for all $\varkappa \geq 0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the inequality*

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{W} \varphi \| \leq c^* ||| \widehat{W} |||_{\gamma, M} \| \widehat{G}_-^{\frac{1}{2}} \varphi \|$$

holds.

Proof. From (0.11) (for $\varepsilon = 1$) it follows that for all $\varkappa \geq 0$, all $k \in \mathbb{R}^d$, and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the estimate

$$\| \widehat{W} \varphi \| \leq ||| \widehat{W} |||_{\gamma, M} \left(v^{\frac{1}{2}}(K) \left(\sum_{N \in \Lambda^*} |k_{\parallel} + 2\pi N_{\parallel}|^2 \|\varphi_N\|^2 \right)^{\frac{1}{2}} + C'(\gamma, 1) \|\varphi\| \right) \leq \quad (4.1)$$

$$\leq ||| \widehat{W} |||_{\gamma, M} \left(\| \widehat{G}_-^1 \varphi \| + C'(\gamma, 1) \|\varphi\| \right)$$

holds. For any vector function $\psi \in L^2(K; \mathbb{C}^M)$, we have $\widehat{G}_-^{-1} \psi \in \widetilde{H}^1(K; \mathbb{C}^M)$. Therefore, from (4.1) (taking into account that $G_N^-(k; \varkappa) \geq \pi |\gamma|^{-1}$, $N \in \Lambda^*$, if $|(k, \gamma)| = \pi$) for all $\varkappa \geq 0$, all $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all $\psi \in L^2(K; \mathbb{C}^M)$ we obtain

$$\| \widehat{W} \widehat{G}_-^{-1} \psi \| \leq c^* ||| \widehat{W} |||_{\gamma, M} \|\psi\|, \quad (4.2)$$

where $c^* = 1 + \pi^{-1} |\gamma| C'(\gamma, 1)$. The same inequality is fulfilled for the adjoint matrix function $\widehat{W}^* \in L^2(K; \mathcal{M}_M)$ (for which $||| \widehat{W}^* |||_{\gamma, M} = ||| \widehat{W} |||_{\gamma, M} < +\infty$). Hence, for the operator $(\widehat{W}^* \widehat{G}_-^{-1})^*$ which is adjoint to the operator $\widehat{W}^* \widehat{G}_-^{-1}$, estimate (4.2) is also satisfied. Since $(\widehat{W}^* \widehat{G}_-^{-1})^* \psi = \widehat{G}_-^{-1} \widehat{W} \psi$ for all vector functions $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$, we conclude that for such vector functions ψ the following inequality is also valid:

$$\| \widehat{G}_-^{-1} \widehat{W} \psi \| \leq c^* ||| \widehat{W} |||_{\gamma, M} \|\psi\|. \quad (4.3)$$

Let now use the interpolation of the operators $\widehat{W} \widehat{G}_-^{-1}$ and $\widehat{G}_-^{-1} \widehat{W}$ (see [47, 3]). Consider the analytic operator function $\zeta \rightarrow \widehat{G}_-^{\zeta} \widehat{W} \widehat{G}_-^{-1-\zeta}$ defined for $\zeta \in \mathbb{C}$ with $-1 < \operatorname{Re} \zeta < 0$. For a vector function $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$, the function $\zeta \rightarrow \widehat{G}_-^{\zeta} \widehat{W} \widehat{G}_-^{-1-\zeta} \psi \in L^2(K; \mathbb{C}^M)$ is also analytic. Moreover, it has continuous bounded extension to the closed set $\{\zeta \in \mathbb{C} : -1 \leq \operatorname{Re} \zeta \leq 0\}$. If either $\operatorname{Re} \zeta = 0$ or $\operatorname{Re} \zeta = -1$, then from (4.2) and (4.3) we get

$$\| \widehat{G}_-^{\zeta} \widehat{W} \widehat{G}_-^{-1-\zeta} \psi \| \leq c^* ||| \widehat{W} |||_{\gamma, M} \|\psi\|. \quad (4.4)$$

Therefore estimate (4.4) is true for all $\zeta \in \mathbb{C}$ with $-1 \leq \operatorname{Re} \zeta \leq 0$. In particular, for $\zeta = -\frac{1}{2}$ (and for $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$) we obtain

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{W} \widehat{G}_-^{-\frac{1}{2}} \psi \| \leq c^* ||| \widehat{W} |||_{\gamma, M} \|\psi\|. \quad (4.5)$$

By continuity, inequality (4.5) holds for all vector functions $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$. Finally, since any vector function $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ can be represented in the form $\varphi = \widehat{G}_-^{-\frac{1}{2}} \psi$, where $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$, the inequality claimed in Lemma 4.1 is an immediate consequence of inequality (4.5). \square

Proof of Theorem 1.4. Fix a number $\delta \in (0, 1)$. Let us choose a sufficiently large number $R > 0$ such that

$$||| \widehat{W} - \widehat{W}_{\{R\}} |||_{\gamma, M}^2 \leq \frac{1}{18} \delta^2 (c^*)^{-2} C_2^2,$$

where $c^* = c^*(\gamma) > 0$ is the constant from Lemma 4.1. The functions $\widehat{V}_{\{R\}}^{(s)}$, $s = 0, 1$, and $A_{\{R\}}$ satisfy the conditions of Theorem 1.3, therefore there are numbers $\tilde{a} = \tilde{a}(C_2; \frac{\delta}{2}, R) \in (0, C_2]$ and $\varkappa'_0 > 0$ such that for all $\varkappa \geq \varkappa'_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$ the inequality

$$\|(\widehat{P}_*^+ + \tilde{a}\widehat{P}_*^-)(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}})\varphi\|^2 \geq (1 - \frac{\delta}{2}) (C_2^2 \|\widehat{G}_-^1 \widehat{P}^- \varphi\|^2 + \tilde{a}^2 \|\widehat{G}_+^1 \widehat{P}^+ \varphi\|^2) \quad (4.6)$$

holds. Instead of the vector $\gamma \in \Lambda \setminus \{0\}$, in conditions (\mathbf{A}_1) , $(\tilde{\mathbf{A}}_1)$, and (\mathbf{A}_2) we can pick the vector $-\gamma$. Under the replacement of the vector γ by the vector $-\gamma$, the measure μ and the constants C_2 and \tilde{a} do not change. The number \varkappa'_0 (which coincides with the number \varkappa_0 from Theorem 1.3 and is determined by the number $\frac{\delta}{2}$ and the function $\widehat{W}_{\{R\}}$) do not change as well. Nevertheless, the orthogonal projections \widehat{P}^+ and \widehat{P}^- are replaced by the orthogonal projections \widehat{P}_*^- and \widehat{P}_*^+ , respectively. Therefore, along with inequality (4.6), the inequality

$$\|(\widehat{P}^- + \tilde{a}\widehat{P}^+)(\widehat{\mathcal{D}}(k - i\varkappa e) + \widehat{W}_{\{R\}})\varphi\|^2 \geq (1 - \frac{\delta}{2}) (C_2^2 \|\widehat{G}_-^1 \widehat{P}_*^+ \varphi\|^2 + \tilde{a}^2 \|\widehat{G}_+^1 \widehat{P}_*^- \varphi\|^2) \quad (4.7)$$

holds. Since $\widehat{W} \in \mathbb{L}_M^\Lambda(d; 0)$ (hence, also $\widehat{W}_{\{R\}} \in \mathbb{L}_M^\Lambda(d; 0)$), the operators $\widehat{\mathcal{D}}(k \pm i\varkappa e) + \widehat{W}_{\{R\}}$ are closed and from (4.6), (4.7) it follows that their ranges $R(\widehat{\mathcal{D}}(k \pm i\varkappa e) + \widehat{W}_{\{R\}})$ are closed subspaces in $L^2(K; \mathbb{C}^M)$. We have

$$(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}})^* = \widehat{\mathcal{D}}(k - i\varkappa e) + \widehat{W}_{\{R\}}.$$

Consequently, from (4.6), (4.7) we see that

$$\text{Ker}(\widehat{\mathcal{D}}(k \pm i\varkappa e) + \widehat{W}_{\{R\}}) = \text{Coker}(\widehat{\mathcal{D}}(k \pm i\varkappa e) + \widehat{W}_{\{R\}}) = \{0\}.$$

The last equalities mean that the operators $\widehat{\mathcal{D}}(k \pm i\varkappa e) + \widehat{W}_{\{R\}}$ (for $\varkappa \geq \varkappa'_0$) are bijective maps of $D(\widehat{\mathcal{D}}(k \pm i\varkappa e) + \widehat{W}_{\{R\}}) = \tilde{H}^1(K; \mathbb{C}^M)$ onto $L^2(K; \mathbb{C}^M)$. From this, using (4.6), (4.7), for all $\varkappa \geq \varkappa'_0$, all $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all $\psi \in L^2(K; \mathbb{C}^M)$ we obtain

$$\|(C_2 \widehat{G}_-^1 \widehat{P}^- + \tilde{a} \widehat{G}_+^1 \widehat{P}^+)(\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}})^{-1}(\widehat{P}_*^+ + \tilde{a}^{-1} \widehat{P}_*^-)\psi\|^2 \leq \quad (4.8)$$

$$\leq (1 - \frac{\delta}{2})^{-1} \|\psi\|^2,$$

$$\|(C_2 \widehat{G}_-^1 \widehat{P}_*^+ + \tilde{a} \widehat{G}_+^1 \widehat{P}_*^-)(\widehat{\mathcal{D}}(k - i\varkappa e) + \widehat{W}_{\{R\}})^{-1}(\widehat{P}^- + \tilde{a}^{-1} \widehat{P}^+)\psi\|^2 \leq \quad (4.9)$$

$$\leq (1 - \frac{\delta}{2})^{-1} \|\psi\|^2.$$

Estimate (4.9) is also valid for the adjoint operator

$$\left((C_2 \widehat{G}_-^1 \widehat{P}_*^+ + \widetilde{a} \widehat{G}_+^1 \widehat{P}_*^-) (\widehat{\mathcal{D}}(k - i\kappa e) + \widehat{W}_{\{R\}})^{-1} (\widehat{P}^- + \widetilde{a}^{-1} \widehat{P}^+) \right)^*,$$

hence, for all $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$,

$$\begin{aligned} \left\| (\widehat{P}^- + \widetilde{a}^{-1} \widehat{P}^+) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_{\{R\}})^{-1} (C_2 \widehat{G}_-^1 \widehat{P}_*^+ + \widetilde{a} \widehat{G}_+^1 \widehat{P}_*^-) \psi \right\|^2 &\leq \\ &\leq \left(1 - \frac{\delta}{2}\right)^{-1} \|\psi\|^2. \end{aligned} \quad (4.10)$$

Let us now use the interpolation of operators (see [47, 3] and also the proof of Lemma 4.1). From (4.8) and (4.10) it follows that for all $\zeta \in \mathbb{C}$ with $0 \leq \operatorname{Re} \zeta \leq 1$ (and all $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$) the inequality

$$\begin{aligned} \left\| (C_2^\zeta \widehat{G}_-^\zeta \widehat{P}^- + \widetilde{a}^{2\zeta-1} \widehat{G}_+^\zeta \widehat{P}^+) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_{\{R\}})^{-1} (C_2^{1-\zeta} \widehat{G}_-^{1-\zeta} \widehat{P}_*^+ + \widetilde{a}^{1-2\zeta} \widehat{G}_+^{1-\zeta} \widehat{P}_*^-) \psi \right\|^2 &\leq \\ &\leq \left(1 - \frac{\delta}{2}\right)^{-1} \|\psi\|^2 \end{aligned}$$

is fulfilled (we also use the uniform boundedness on the closed set $\{\zeta \in \mathbb{C} : 0 \leq \operatorname{Re} \zeta \leq 1\}$ of the vector function from the left hand side of the last inequality; it follows easily from (4.8)). For $\zeta = \frac{1}{2}$, the last inequality has the form

$$\begin{aligned} \left\| (C_2^{\frac{1}{2}} \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- + \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_{\{R\}})^{-1} (C_2^{\frac{1}{2}} \widehat{G}_-^{\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{\frac{1}{2}} \widehat{P}_*^-) \psi \right\|^2 &\leq \\ &\leq \left(1 - \frac{\delta}{2}\right)^{-1} \|\psi\|^2. \end{aligned} \quad (4.11)$$

By continuity (see (4.8)), estimate (4.11) is true for all vector functions $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$. Since any vector function $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ can be represented in the form

$$\varphi = (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_{\{R\}})^{-1} (C_2^{\frac{1}{2}} \widehat{G}_-^{\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{\frac{1}{2}} \widehat{P}_*^-) \psi,$$

where $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$, from (4.11) it follows that (for all $\kappa \geq \kappa'_0$, all $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$) the inequality

$$\begin{aligned} \left\| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{\mathcal{D}}(k + i\kappa e) + \widehat{W}_{\{R\}}) \varphi \right\|^2 &\geq \\ &\geq \left(1 - \frac{\delta}{2}\right) \left\| (C_2^{\frac{1}{2}} \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- + \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+) \varphi \right\|^2 \end{aligned}$$

holds. Finally, using the estimate

$$\begin{aligned} \left\| \widehat{G}_-^{-\frac{1}{2}} (\widehat{W} - \widehat{W}_{\{R\}}) \varphi \right\|^2 &\leq (c^*)^2 \left\| \widehat{W} - \widehat{W}_{\{R\}} \right\|_{\gamma, M}^2 \left\| \widehat{G}_-^{\frac{1}{2}} \varphi \right\|^2 \leq \\ &\leq \frac{1}{18} \delta^2 C_2^2 \left\| \widehat{G}_-^{\frac{1}{2}} \varphi \right\|^2 \leq \frac{1}{18} \delta^2 C_2^2 \left(\left\| \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi \right\|^2 + \left\| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi \right\|^2 \right) \end{aligned}$$

which is a consequence of Lemma 4.1, for all $\varkappa \geq \varkappa'_0$, all vectors $k \in \mathbb{R}^d$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$ we obtain the inequality

$$\begin{aligned}
& \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}) \varphi \|^2 \geq \\
& \geq \left(1 - \frac{\delta}{3}\right) \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{\mathcal{D}}(k + i\varkappa e) + \widehat{W}_{\{R\}}) \varphi \|^2 - \\
& - \frac{3}{\delta} \| (C_2^{-\frac{1}{2}} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ + \widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^-) (\widehat{W} - \widehat{W}_{\{R\}}) \varphi \|^2 \geq \\
& \geq \left(1 - \frac{5\delta}{6}\right) \| (C_2^{\frac{1}{2}} \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- + \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+) \varphi \|^2 - \frac{3}{\delta} C_2^{-1} \| \widehat{G}_-^{-\frac{1}{2}} (\widehat{W} - \widehat{W}_{\{R\}}) \varphi \|^2 \geq \\
& \geq \left(1 - \frac{5\delta}{6}\right) (C_2 \| \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi \|^2 + \| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi \|^2) - \frac{\delta}{6} C_2 (\| \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi \|^2 + \| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi \|^2) \geq \\
& \geq (1 - \delta) (C_2 \| \widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi \|^2 + \| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi \|^2).
\end{aligned}$$

Theorem 1.4 is proved.

5 Proof of Theorem 1.5

In this Section we shall use the modification of the method suggested in [34].

Let $\#\mathcal{O}$ denote the number of elements of a finite set \mathcal{O} . By $d(K^*)$ denote the diameter of the fundamental domain K^* . For a measurable matrix function $\widehat{W} : K \rightarrow \mathcal{M}_M$ and a number $a \geq 0$, we write

$$\widehat{W}_{[a]}(x) = \begin{cases} \widehat{W}(x) & \text{if } \|\widehat{W}(x)\| \geq a, \\ \widehat{0} & \text{otherwise.} \end{cases}$$

If $\widehat{W} = (W_{pq})_{p,q=1,\dots,M} \in L^2(K; \mathcal{M}_M)$, then

$$\begin{aligned}
& \sum_{N \in \Lambda^*} \|\widehat{W}_N\|^2 \leq \sum_{N \in \Lambda^*} \sum_{p,q=1}^M |(W_{pq})_N|^2 = \\
& = v^{-1}(K) \int_K \left(\sum_{p,q=1}^M |W_{pq}(x)|^2 \right) dx \leq M v^{-1}(K) \int_K \|\widehat{W}(x)\|^2 dx.
\end{aligned} \tag{5.1}$$

First we shall obtain some auxiliary statements which will be used in the proof of Theorems 1.5 and 1.6.

As above we suppose that the vector $\gamma \in \Lambda \setminus \{0\}$ is fixed, $e = |\gamma|^{-1} \gamma$. Choose a number $\Xi \geq 1$ (we set $\Xi > 1$ in this Section and $\Xi = 1$ in Section 6). Let $\widehat{V} = \widehat{V}^{(0)} + \widehat{V}^{(1)}$, where $\widehat{V}^{(s)} \in L_w^3(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$. For the Fourier coefficients $\widehat{V}_N^{(s)}$, $N \in \Lambda^*$, of the matrix functions $\widehat{V}^{(s)}$, $s = 0, 1$, we have $\widehat{V}_N^{(s)} \in \mathcal{S}_M^{(s)}$. Hence, the Fourier coefficients $\widehat{V}_N^{(s)}$ (as operators acting on \mathbb{C}^M) commute with all orthogonal projections $\widehat{P}_{\tilde{e}}^\pm$, $\tilde{e} \in S_1(e)$. The same assertion is true for the Fourier coefficients \widehat{V}_N , $N \in \Lambda^*$.

Let us choose (and fix) a number $h \in \mathbb{R}$ for which $h \geq 64$ and $h > 2\pi d(K^*)$. The number $\varkappa_0 > 0$ is chosen sufficiently large (lower estimates for the number \varkappa_0 are specified in the sequel). To start with, we assume that $\varkappa_0 > 2h$. Let $\varkappa_0 \leq \varkappa_1 \leq \varkappa \leq \Xi \varkappa_1$ (if we set $\Xi = 1$, then $\varkappa_1 = \varkappa$), $l = l(h, \varkappa_1) \in \mathbb{N}$ is the largest integer with $2h^l \leq \varkappa_1$ (hence, $\varkappa_1 < 2h^{l+1}$ and $2h^l \leq \varkappa < 2\Xi h^{l+1}$).

In what follows, the vector $k \in \mathbb{R}^3$ is assumed to satisfy the condition $|(k, \gamma)| = \pi$. For any $b \in [0, \varkappa)$, we write

$$\mathcal{K}(b) = \{N \in \Lambda^* : G_N(k; \varkappa) \leq b\}$$

(the sets $\mathcal{K}(b)$ depend also on k and \varkappa). For $b > 2\pi d(K^*)$ (and $b < \varkappa$) the following estimate holds:

$$\#\mathcal{K}(b) \leq c_2 v^{-1}(K^*) \varkappa b^2,$$

where $c_2 > 0$ is a universal constant. If $N \in \mathcal{K}(b)$, then $k_\perp + 2\pi N_\perp \neq 0$. If $N \in \mathcal{K}(h^l)$, then

$$|\varkappa - |k_\perp + 2\pi N_\perp|| \leq h^l \leq \frac{\varkappa}{2},$$

hence, $|k_\perp + 2\pi N_\perp| \geq \frac{\varkappa}{2} > 1$. From this it follows that for all $N, N' \in \mathcal{K}(h^l)$,

$$|\tilde{e}(k + 2\pi N) - \tilde{e}(k + 2\pi N')| \leq \frac{4\pi}{\varkappa} |N_\perp - N'_\perp|. \quad (5.2)$$

We introduce the notation

$$\mathcal{K}_1 \doteq \mathcal{K}(h), \quad \mathcal{K}_\mu \doteq \{N \in \Lambda^* : h^{\mu-1} < G_N(k; \varkappa) \leq h^\mu\}, \quad \mu = 2, \dots, l.$$

For any vector function $\varphi \in L^2(K; \mathbb{C}^M)$,

$$\sqrt{\frac{\pi}{|\gamma|}} \|\hat{P}^{\mathcal{K}_1} \varphi\| \leq \|\hat{G}_-^{\frac{1}{2}} \hat{P}^{\mathcal{K}_1} \varphi\| \leq h^{\frac{1}{2}} \|\hat{P}^{\mathcal{K}_1} \varphi\|, \quad (5.3)$$

$$h^{\frac{\mu-1}{2}} \|\hat{P}^{\mathcal{K}_\mu} \varphi\| \leq \|\hat{G}_-^{\frac{1}{2}} \hat{P}^{\mathcal{K}_\mu} \varphi\| \leq h^{\frac{\mu}{2}} \|\hat{P}^{\mathcal{K}_\mu} \varphi\|, \quad \mu = 2, \dots, l. \quad (5.4)$$

Let us denote

$$\hat{P}_\mu^{(\pm)} \doteq \hat{P}^\pm(k; e) \hat{P}^{\mathcal{K}_\mu}, \quad \mu = 1, \dots, l, \quad \hat{P}^{(\pm)} \doteq \hat{P}^\pm(k; e) \hat{P}^{\mathcal{K}(h^l)} = \sum_{\mu=1}^l \hat{P}_\mu^{(\pm)}.$$

The equality $\hat{P}^{(+)} + \hat{P}^{(-)} = \hat{P}^{\mathcal{K}(h^l)}$ holds.

For all $\varkappa > 0$ and all $N \in \Lambda^*$, we get

$$G_N^+(k; \varkappa) > |k + 2\pi N| \geq |k_\parallel + 2\pi N_\parallel| \geq \pi |\gamma|^{-1}.$$

Using the inequality $\varkappa < 2\Xi h^{l+1}$, for all $N \in \Lambda^* \setminus \mathcal{K}(h^l)$, we obtain

$$G_N^-(k; \varkappa) > h^l (2\varkappa + h^l)^{-1} G_N^+(k; \varkappa) > (4\Xi h + 1)^{-1} G_N^+(k; \varkappa), \quad (5.5)$$

$$G_N^-(k; \varkappa) > h^l (\varkappa + h^l)^{-1} |k + 2\pi N| > (2\Xi h + 1)^{-1} |k + 2\pi N|. \quad (5.6)$$

Lemma 5.1. Suppose $\widehat{W} \in L^2(K; \mathcal{M}_M)$. Then for any finite set $\mathcal{O} \subset \Lambda^*$ and any vector function $\varphi \in L^2(K; \mathbb{C}^M)$ we have

$$\|\widehat{W}\widehat{P}^{\mathcal{O}}\varphi\| \leq v^{-\frac{1}{2}}(K) (\#\mathcal{O})^{\frac{1}{2}} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} \|\widehat{P}^{\mathcal{O}}\varphi\|.$$

Proof. Indeed,

$$\begin{aligned} \|\widehat{W}\widehat{P}^{\mathcal{O}}\varphi\| &\leq \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} \|\widehat{P}^{\mathcal{O}}\varphi\|_{L^\infty(K; \mathbb{C}^M)} \leq \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} \left(\sum_{N \in \mathcal{O}} \|\varphi_N\| \right) \leq \\ &\leq (\#\mathcal{O})^{\frac{1}{2}} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} \left(\sum_{N \in \mathcal{O}} \|\varphi_N\|^2 \right)^{\frac{1}{2}} = v^{-\frac{1}{2}}(K) (\#\mathcal{O})^{\frac{1}{2}} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} \|\widehat{P}^{\mathcal{O}}\varphi\|. \end{aligned}$$

□

Lemma 5.2. Suppose $\widehat{W} \in L_w^3(K; \mathcal{M}_M)$. Then there exists a nonincreasing function $[0, +\infty) \ni t \rightarrow f_{\widehat{W}}(t) \in [0, +\infty)$ such that

$$f_{\widehat{W}}(t) \rightarrow \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}$$

as $t \rightarrow +\infty$, and for any finite set $\mathcal{O} \subset \Lambda^*$ and any vector function $\varphi \in L^2(K; \mathbb{C}^M)$ the inequality

$$\|\widehat{W}\widehat{P}^{\mathcal{O}}\varphi\| \leq 4v^{-\frac{1}{3}}(K) (\#\mathcal{O})^{\frac{1}{3}} f_{\widehat{W}}(\#\mathcal{O}) \|\widehat{P}^{\mathcal{O}}\varphi\|$$

holds.

Proof. For $a > 0$, we get

$$\begin{aligned} \|\widehat{W}_{[a]}\|_{L^2(K; \mathcal{M}_M)}^2 &= \sum_{\nu=1}^{+\infty} \int_{\{x \in K : 2^{\nu-1}a \leq \|\widehat{W}(x)\| < 2^\nu a\}} \|\widehat{W}(x)\|^2 dx \leq \\ &\leq \frac{8}{a} \sum_{\nu=1}^{+\infty} 2^{-\nu} \|\widehat{W}_{[2^{\nu-1}a]}\|_{L_w^3(K; \mathcal{M}_M)}^3 \leq \frac{8}{a} \|\widehat{W}_{[a]}\|_{L_w^3(K; \mathcal{M}_M)}^3. \end{aligned}$$

Hence, for all vector functions $\varphi \in L^2(K; \mathbb{C}^M)$,

$$\begin{aligned} \|\widehat{W}\widehat{P}^{\mathcal{O}}\varphi\| &\leq \|(\widehat{W} - \widehat{W}_{[a]})\widehat{P}^{\mathcal{O}}\varphi\| + \|\widehat{W}_{[a]}\widehat{P}^{\mathcal{O}}\varphi\| \leq \\ &\leq (a + v^{-\frac{1}{2}}(K) (\#\mathcal{O})^{\frac{1}{2}} \|\widehat{W}_{[a]}\|_{L^2(K; \mathcal{M}_M)}) \|\widehat{P}^{\mathcal{O}}\varphi\| \leq \\ &\leq \left(a + \frac{2\sqrt{2}}{\sqrt{a}} v^{-\frac{1}{2}}(K) (\#\mathcal{O})^{\frac{1}{2}} \|\widehat{W}_{[a]}\|_{L^3(K; \mathcal{M}_M)}^{\frac{3}{2}} \right) \|\widehat{P}^{\mathcal{O}}\varphi\| \end{aligned} \tag{5.7}$$

(see Lemma 5.1). Fix a number $\varepsilon > 0$. We can represent the fundamental domain K in the form $\bigcup_{j=1}^J K_j^{(\varepsilon)}$, where $J \in \mathbb{N}$, and $K_j^{(\varepsilon)}$ are disjoint measurable sets of sufficiently small diameters $d(K_j^{(\varepsilon)})$ such that for all j the following inequalities are fulfilled:

$$\|\chi_{K_j^{(\varepsilon)}} \widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty)} < \frac{\varepsilon}{2} + \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}$$

(here χ_T is the characteristic function of a set $T \subseteq K$). For every $j = 1, \dots, J$, we choose a number $a_j > 0$ such that

$$\|\chi_{K_j^{(\varepsilon)}} \widehat{W}_{[a_j]}\|_{L_w^3(K; \mathcal{M}_M)} < \frac{\varepsilon}{2} + \|\chi_{K_j^{(\varepsilon)}} \widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty)} < \varepsilon + \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}.$$

From (5.7), for $a \geq \max_j a_j$, we derive

$$\begin{aligned} \|\widehat{W} \widehat{P}^{\mathcal{O}} \varphi\|^2 &= \sum_j \|\chi_{K_j^{(\varepsilon)}} \widehat{W} \widehat{P}^{\mathcal{O}} \varphi\|^2 \leq \\ &\leq 2 \sum_j \left(a^2 + \frac{8}{a} v^{-1}(K) (\# \mathcal{O}) \|\chi_{K_j^{(\varepsilon)}} \widehat{W}_{[a_j]}\|_{L_w^3(K; \mathcal{M}_M)}^3 \right) \|\chi_{K_j^{(\varepsilon)}} \widehat{P}^{\mathcal{O}} \varphi\|^2 \leq \\ &\leq 2 \left(a^2 + \frac{8}{a} v^{-1}(K) (\# \mathcal{O}) (\varepsilon + \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})})^3 \right) \sum_j \|\chi_{K_j^{(\varepsilon)}} \widehat{P}^{\mathcal{O}} \varphi\|^2. \end{aligned} \quad (5.8)$$

If

$$\# \mathcal{O} \geq \frac{1}{8} v(K) (\varepsilon + \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})})^{-3} \max_j a_j^3, \quad (5.9)$$

then we put

$$a = 2 v^{-\frac{1}{3}}(K) (\# \mathcal{O})^{\frac{1}{3}} (\varepsilon + \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}).$$

Therefore inequality (5.8) (under condition (5.9)) implies that

$$\|\widehat{W} \widehat{P}^{\mathcal{O}} \varphi\| \leq 4 v^{-\frac{1}{3}}(K) (\# \mathcal{O})^{\frac{1}{3}} (\varepsilon + \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}) \|\widehat{P}^{\mathcal{O}} \varphi\|. \quad (5.10)$$

Since the number $\varepsilon > 0$ can be chosen as small as we wish, the inequality claimed in Lemma 5.2 is a consequence of (5.10). \square

Lemma 5.3. *There is a number $c_3 = c_3(h) > 0$ such that for any $\varepsilon > 0$ and any matrix function $\widehat{V} \in L_w^3(K; \mathcal{M}_M)$, for which*

$$\|\widehat{V}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \leq c_3 \varepsilon, \quad (5.11)$$

there exists a number $\varkappa'_0(\varepsilon) = \varkappa'_0(\varepsilon; \Lambda, \gamma, h, \widehat{V}) > 2h$ such that for all $\varkappa_1 \geq \varkappa'_0(\varepsilon)$, all $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in L^2(K; \mathbb{C}^M)$ the inequality

$$\|\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{\mathcal{K}(h^l)} \varphi\| \leq \varepsilon \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\mathcal{K}(h^l)} \varphi\| \quad (5.12)$$

holds.

Proof. Taking into account the estimates $\# \mathcal{K}_\mu \leq c_2 v^{-1}(K^*) \varkappa h^{2\mu}$, $\mu = 1, \dots, l$, and

$$h^{\frac{1}{2}} \|\widehat{P}^{\mathcal{K}_1} \varphi\| \leq \sqrt{\frac{|\gamma| h}{\pi}} \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\mathcal{K}_1} \varphi\|, \quad h^{\frac{\mu}{2}} \|\widehat{P}^{\mathcal{K}_\mu} \varphi\| \leq h^{\frac{1}{2}} \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\mathcal{K}_\mu} \varphi\|, \quad \mu = 2, \dots, l,$$

and using Lemma 5.2, for all vector functions $\varphi \in L^2(K; \mathbb{C}^M)$ we obtain

$$\begin{aligned}
& \|\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{\mathcal{K}(h^l)} \varphi\| \leq \varkappa^{-\frac{1}{2}} \sum_{\mu=1}^l \|\widehat{V} \widehat{P}^{\mathcal{K}_\mu} \varphi\| \leq \\
& \leq 4 \varkappa^{-\frac{1}{2}} v^{-\frac{1}{3}}(K) \sum_{\mu=1}^l (c_2 v^{-1}(K^*) \varkappa h^{2\mu})^{\frac{1}{3}} f_{\widehat{V}}(c_2 v^{-1}(K^*) \varkappa h^{2\mu}) \|\widehat{P}^{\mathcal{K}_\mu} \varphi\| \leq \\
& \leq 4 c_2^{\frac{1}{3}} f_{\widehat{V}}(c_2 v^{-1}(K^*) \varkappa h^2) \varkappa^{-\frac{1}{6}} \sum_{\mu=1}^l h^{\frac{\mu}{6}} (h^{\frac{\mu}{2}} \|\widehat{P}^{\mathcal{K}_\mu} \varphi\|) \leq \\
& \leq 4 c_2^{\frac{1}{3}} h^{\frac{1}{2}} \left(\sqrt{\frac{|\gamma|}{\pi}} \varkappa^{-\frac{1}{6}} h^{\frac{1}{6}} + \varkappa^{-\frac{1}{6}} \sum_{\mu=2}^l h^{\frac{\mu}{6}} \right) f_{\widehat{V}}(c_2 v^{-1}(K^*) \varkappa h^2) \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{\mathcal{K}(h^l)} \varphi\|.
\end{aligned} \tag{5.13}$$

Since

$$\varkappa^{-\frac{1}{6}} \sum_{\mu=2}^l h^{\frac{\mu}{6}} = (\varkappa^{-1} h^l)^{\frac{1}{6}} \sum_{\nu=0}^{l-2} h^{-\frac{\nu}{6}} < 2^{\frac{5}{6}} < 2$$

and

$$f_{\widehat{V}}(c_2 v^{-1}(K^*) \varkappa h^2) \rightarrow \|\widehat{V}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}$$

as $\varkappa \rightarrow +\infty$, estimate (5.12) follows from (5.13) for

$$c_3 = \frac{1}{8} c_2^{-\frac{1}{3}} h^{-\frac{1}{2}}$$

and under the choice of a sufficiently large number $\varkappa'_0(\varepsilon)$. \square

In the sequel, we fix a number $\varepsilon' > 0$ and put $\varepsilon = \varepsilon'/3$, $\varkappa_0 \geq \varkappa'_0(\varepsilon)$. We assume that condition (5.11) is fulfilled for the function $\widehat{V} \in L_w^3(K; \mathcal{M}_M)$. From (5.12) it follows that

$$\|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}^{(-)} \widehat{V} \widehat{P}^{(-)} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(-)} \varphi\|, \tag{5.14}$$

$$\|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}^{(-)} \widehat{V} \widehat{P}^{(+)} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(+)} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(+)} \varphi\|. \tag{5.15}$$

We shall also assume that $\varkappa_0 \geq \varkappa'_0((4\Xi h + 1)^{-\frac{1}{2}} \varepsilon)$. Then (5.5) and (5.12) yield

$$\|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} \widehat{P}^{(-)} \varphi\| \leq \tag{5.16}$$

$$\leq (4\Xi h + 1)^{\frac{1}{2}} \|\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{(-)} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(-)} \varphi\|,$$

$$\|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} \widehat{P}^{(+)} \varphi\| \leq \tag{5.17}$$

$$\leq (4\Xi h + 1)^{\frac{1}{2}} \|\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{(+)} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(+)} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(+)} \varphi\|.$$

If we put $\varphi = \widehat{G}_-^{-\frac{1}{2}} \psi$, where $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$, then (5.12) implies that

$$\| \widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{\mathcal{K}(h^l)} \widehat{G}_-^{-\frac{1}{2}} \psi \| \leq \varepsilon \| P^{\mathcal{K}(h^l)} \psi \| \leq \varepsilon \| \psi \|.$$

By continuity, the operator $\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{\mathcal{K}(h^l)} \widehat{G}_-^{-\frac{1}{2}}$ having the domain

$$D(\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{P}^{\mathcal{K}(h^l)} \widehat{G}_-^{-\frac{1}{2}}) = \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$$

extends to the bounded operator $\widehat{G}_+^{-\frac{1}{2}} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\mathcal{K}(h^l)}$ acting on the space $L^2(K; \mathbb{C}^M)$. Therefore, for the adjoint operator that takes a vector function $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$ to the vector function $\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\mathcal{K}(h^l)} \widehat{V} \widehat{G}_+^{-\frac{1}{2}} \psi$, the estimate

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\mathcal{K}(h^l)} \widehat{V} \widehat{G}_+^{-\frac{1}{2}} \psi \| \leq \varepsilon \| \psi \|, \quad \psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M), \quad (5.18)$$

is fulfilled as well. If we put $\psi = \widehat{G}_+^{\frac{1}{2}} \varphi$, where $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$, then (5.18) yields

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\mathcal{K}(h^l)} \widehat{V} \varphi \| \leq \varepsilon \| \widehat{G}_+^{\frac{1}{2}} \varphi \|. \quad (5.19)$$

In particular, for any vector function $\varphi \in L^2(K; \mathbb{C}^M)$,

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{V} \widehat{P}^{(+)} \varphi \| \leq \varepsilon \| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(+)} \varphi \|. \quad (5.20)$$

Since we assume that $\varkappa_0 \geq \varkappa'_0 ((4\Xi h + 1)^{-\frac{1}{2}} \varepsilon)$, inequality (5.19) is also true under the change ε to $(4\Xi h + 1)^{-\frac{1}{2}} \varepsilon$. Therefore (see (5.5)) for any vector function $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ we get

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \| \leq \quad (5.21)$$

$$\begin{aligned} &\leq (4\Xi h + 1)^{-\frac{1}{2}} \varepsilon \| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \| \leq \varepsilon \| \widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \|, \\ &\| \widehat{G}_+^{-\frac{1}{2}} \widehat{P}^{(-)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \| \leq \| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \| \leq \\ &\leq (4\Xi h + 1)^{-\frac{1}{2}} \varepsilon \| \widehat{G}_+^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \| \leq \varepsilon \| \widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi \|. \end{aligned} \quad (5.22)$$

Now, suppose that the condition

$$\| \widehat{V} \|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} < C^{-1} (2\Xi h + 1)^{-1} \varepsilon \quad (5.23)$$

is satisfied along with condition (5.11), where $C = C(3) > 0$ is the constant from (0.6). From (0.6), (1.1), and (5.6) it follows that under the choice of a sufficiently large number $\varkappa_0'' > 2h$ (dependent on $\varepsilon, \Lambda, \gamma, \widehat{V}, \Xi, h$) and for all $\varkappa_1 \geq \varkappa_0 \geq \varkappa_0''$, all $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$ the inequality

$$\| \widehat{V} \widehat{G}_-^{-1} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi \| \leq \varepsilon \| \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi \| \quad (5.24)$$

holds (and $\widehat{G}_-^{-1}\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$). Furthermore,

$$(\widehat{V} \widehat{G}_-^{-1} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)})^* \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi = \widehat{G}_-^{-1} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi$$

for the adjoint operator $(\widehat{V} \widehat{G}_-^{-1} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)})^*$ and all vector functions $\psi \in \widetilde{H}^1(K; \mathbb{C}^M)$, whence

$$\|\widehat{G}_-^{-1} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi\| \leq \varepsilon \|\widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi\|. \quad (5.25)$$

Using the interpolation of operators (see [47, 3] and also the proof of Lemma 4.1), from (5.24) and (5.25), for all vector functions $\psi \in \widetilde{H}^{\frac{1}{2}}(K; \mathbb{C}^M)$ we derive

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi\| \leq \varepsilon \|\widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \psi\|.$$

If we put $\varphi = \widehat{G}_-^{-\frac{1}{2}} \psi \in \widetilde{H}^1(K; \mathbb{C}^M)$, then the previous inequality yields

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\| \leq \varepsilon \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|. \quad (5.26)$$

The following theorem is a key point in the proof of Theorem 1.5.

Theorem 5.1. *Let $\widehat{V}^{(s)} \in L^3 \ln^{1+\delta} L(K; \mathcal{S}_M^{(s)})$, $\delta > 0$, $s = 0, 1$, let $\widehat{V} = \widehat{V}^{(0)} + \widehat{V}^{(1)}$, and let $\Xi > 1$. Then (for a given number $\varepsilon > 0$) there exists a number $\widetilde{\varkappa}_0 > 2h$ such that for any $\varkappa_1 \geq \widetilde{\varkappa}_0$ there is a number $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$ such that for all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in L^2(K; \mathbb{C}^M)$ we have*

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{V} \widehat{P}^{(-)} \varphi\| \leq \varepsilon \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{(-)} \varphi\|. \quad (5.27)$$

Proof. For all $\mu, \nu = 1, \dots, l$ and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$, we shall obtain upper estimates for the norms

$$\|\widehat{P}_\mu^{(+)} \widehat{V} \widehat{P}_\nu^{(-)} \psi\|.$$

Given $\mu, \nu \in \{1, \dots, l\}$ and $n \in \Lambda^*$, by $S_{\mu\nu}(n)$ denote the number of vectors $N \in \mathcal{K}_\mu$ such that $N - n \in \mathcal{K}_\nu$. If either $2\pi|n_\perp| > 2\varkappa + h^\mu + h^\nu$ or $2\pi|n_\parallel| > h^\mu + h^\nu$, then $S_{\mu\nu}(n) = 0$. Since $h > 2\pi d(K^*)$, we have $h^l + 2\pi d(K^*) < 2h^l \leq \varkappa_1 \leq \varkappa$, hence, $h^l < \varkappa - 2\pi d(K^*)$. Under these conditions, there is a universal constant $c_4 > 0$ (see [33]) such that for all $\mu, \nu = 1, \dots, l$ and all vectors $n \in \Lambda^*$ with $\pi|n_\perp| \leq \varkappa + h^{\max\{\mu, \nu\}}$ (and $\pi|n_\parallel| \leq h^{\max\{\mu, \nu\}}$), the estimate

$$S_{\mu\nu}(n) \leq \frac{c_4}{v(K^*)} \frac{h^{\mu+\nu+\min\{\mu, \nu\}} \varkappa^{\frac{3}{2}}}{(\pi|n_\perp| + h^{\max\{\mu, \nu\}}) \sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}}$$

holds. Therefore (also see (1.4) and (5.1)), for all matrix functions $\widehat{W} \in L^2(K; \mathcal{M}_M)$, which have the Fourier coefficients \widehat{W}_N , $N \in \Lambda^*$, that commute with all orthogonal projections \widehat{P}_e^\pm , $\widetilde{e} \in S_1(e)$, and for all vector functions $\psi \in L^2(K; \mathbb{C}^M)$ we get

$$\|\widehat{P}_\mu^{(+)} \widehat{W} \widehat{P}_\nu^{(-)} \psi\|^2 = \quad (5.28)$$

$$\begin{aligned}
&= v(K) \sum_{N \in \mathcal{K}_\mu} \left\| \sum_{\substack{n \in \Lambda^* : \\ N-n \in \mathcal{K}_\nu}} \widehat{P}_{\tilde{e}(k+2\pi N)}^+ \widehat{W}_n \widehat{P}_{\tilde{e}(k+2\pi(N-n))}^- \psi_{N-n} \right\|^2 \leq \\
&\leq \frac{1}{4} v(K) \sum_{N \in \mathcal{K}_\mu} \left(\sum_{\substack{n \in \Lambda^* : \\ N-n \in \mathcal{K}_\nu}} |\tilde{e}(k+2\pi N) - \tilde{e}(k+2\pi(N-n))| \|\widehat{W}_n\| \|(\widehat{P}^- \psi)_{N-n}\| \right)^2 \leq \\
&\leq v(K) \varkappa^{-2} \sum_{N \in \mathcal{K}_\mu} \left(\sum_{\substack{n \in \Lambda^* : \\ N-n \in \mathcal{K}_\nu}} 2\pi |n_\perp| \|\widehat{W}_n\| \|(\widehat{P}^- \psi)_{N-n}\| \right)^2 \leq \\
&\leq v(K) \varkappa^{-2} \sum_{N \in \mathcal{K}_\mu} \left(\sum_{\substack{n \in \Lambda^* : \\ N-n \in \mathcal{K}_\nu}} (2\pi |n_\perp|)^2 \|\widehat{W}_n\|^2 \right) \left(\sum_{\substack{n \in \Lambda^* : \\ N-n \in \mathcal{K}_\nu}} \|(\widehat{P}^- \psi)_{N-n}\|^2 \right) = \\
&= \varkappa^{-2} \left(\sum_{n \in \Lambda^*} \left(\sum_{\substack{N \in \mathcal{K}_\mu : \\ N-n \in \mathcal{K}_\nu}} 1 \right) (2\pi |n_\perp|)^2 \|\widehat{W}_n\|^2 \right) \|\widehat{P}_\nu^{(-)} \psi\|^2 = \\
&= \varkappa^{-2} \left(\sum_{n \in \Lambda^*} S_{\mu\nu}(n) (2\pi |n_\perp|)^2 \|\widehat{W}_n\|^2 \right) \|\widehat{P}_\nu^{(-)} \psi\|^2 \leq \\
&\leq c_4 v(K) h^{\mu+\nu+\min\{\mu,\nu\}} \times \\
&\times \left(\sum_{\substack{n \in \Lambda^* : \\ \pi |n_\perp| \leq \varkappa + h^{\max\{\mu,\nu\}}, \\ \pi |n_\parallel| \leq h^{\max\{\mu,\nu\}}}} \frac{(2\pi |n_\perp|)^2 \|\widehat{W}_n\|^2}{(\pi |n_\perp| + h^{\max\{\mu,\nu\}}) \sqrt{\varkappa(\varkappa + 2h^{\max\{\mu,\nu\}} - \pi |n_\perp|)}} \right) \|\widehat{P}_\nu^{(-)} \psi\|^2.
\end{aligned}$$

For any $a \geq 0$, the Fourier coefficients $(\widehat{V}_{[a]})_N$, $N \in \Lambda^*$, of the matrix function $\widehat{V}_{[a]} \in L^3(K; \mathcal{M}_M) \subset L^2(K; \mathcal{M}_M)$ commute with all orthogonal projections $\widehat{P}_{\tilde{e}}^\pm$, $\tilde{e} \in S_1(e)$. In particular, from this and (5.28), using the inequalities $2\varkappa + 2h^{\max\{\mu,\nu\}} \leq 2\varkappa + 2h^l < 3\varkappa$ and $\varkappa \leq \Xi \varkappa_1$, we obtain

$$\begin{aligned}
&\|\widehat{P}_\mu^{(+)} \widehat{V}_{[a]} \widehat{P}_\nu^{(-)} \psi\| \leq \\
&\leq h^{\frac{1}{2}(\mu+\nu+\min\{\mu,\nu\})} \left(6c_4 \Xi \sqrt{\varkappa_1} v(K) \sum_{\substack{n \in \Lambda^* : \\ \pi |n_\perp| \leq \varkappa + h^{\max\{\mu,\nu\}}}} \frac{\|(\widehat{V}_{[a]})_n\|^2}{\sqrt{\varkappa + 2h^{\max\{\mu,\nu\}} - \pi |n_\perp|}} \right)^{\frac{1}{2}} \|\widehat{P}_\nu^{(-)} \psi\|.
\end{aligned} \tag{5.29}$$

Let $a \geq 2$. Denote

$$Y_\delta(\widehat{V}; a) \doteq \int_{\{x \in K : \|\widehat{V}(x)\| \geq a\}} \|\widehat{V}(x)\|^3 \ln^{1+\delta} \|\widehat{V}(x)\| dx;$$

$Y_\delta(\widehat{V}; a) \downarrow 0$ as $a \rightarrow +\infty$. The following inequality is valid:

$$\begin{aligned}
&\|\widehat{V}_{[a]}\|_{L^2(K; \mathcal{M}_M)}^2 \leq \\
&\leq (a \ln^{1+\delta} a)^{-1} \int_{\{x \in K : \|\widehat{V}(x)\| \geq a\}} \|\widehat{V}(x)\|^3 \ln^{1+\delta} \|\widehat{V}(x)\| dx = (a \ln^{1+\delta} a)^{-1} Y_\delta(\widehat{V}; a).
\end{aligned} \tag{5.30}$$

Choose numbers $r_j \in (0, 1]$, $j \in \mathbb{N}$, such that $r_1 = 1$, $r_j \downarrow 0$, $a_j \doteq h^{\frac{1}{3}j} r_j \uparrow +\infty$ (then $a_j \geq 2$), $r_j^{-1} Y_\delta(\widehat{V}; a_j) \downarrow 0$ as $j \rightarrow +\infty$, and

$$\sum_{j=1}^{+\infty} (\ln a_j)^{-1-\delta} < +\infty.$$

Let us define the functions

$$\mathcal{G}^{(l)}(h; \xi) = \begin{cases} (h - \xi)^{-\frac{1}{2}} & \text{if } \xi < h, \\ (h^k - \xi)^{-\frac{1}{2}} & \text{if } h^{k-1} < \xi < h^k, \quad k = 2, \dots, l, \\ 0 & \text{if } \xi > h^l. \end{cases}$$

For all $\Delta > 0$, we write

$$T_{l,h}(\Delta) = \max_{\tau \in \mathbb{R}} \int_{\tau}^{\tau+\Delta} \mathcal{G}^{(l)}(h; \xi) d\xi.$$

If $0 < \Delta \leq h^2 - h$, then

$$T_{l,h}(\Delta) = \int_0^{\Delta} \frac{d\xi}{\sqrt{\xi}} = 2\sqrt{\Delta}.$$

If either $h^k - h < \Delta \leq h^{k+1} - h$, $k = 2, \dots, l-1$, or $h^l - h < \Delta$ (for $k = l$), then

$$\begin{aligned} T_{l,h}(\Delta) &= \left(\int_0^{h^2-h} + \int_0^{h^3-h^2} + \dots + \int_0^{h^k-h^{k-1}} + \int_0^{\Delta-(h^k-h)} \right) \frac{d\xi}{\sqrt{\xi}} < \\ &< 2 \left(\left(1 - \frac{1}{\sqrt{h}}\right)^{-1} h^{\frac{k}{2}} + \sqrt{\Delta} \right) < 5\sqrt{\Delta}. \end{aligned}$$

Assign the number $j = j(\mu, \nu) = \mu + \nu + \min\{\mu, \nu\} \in \{1, \dots, 3l\}$ to each ordered pair (μ, ν) of numbers $\mu, \nu \in \{1, \dots, l\}$. Let $\mathcal{L}(j)$, where $j = 1, \dots, 3l$, be the set of ordered pairs (μ, ν) with $j(\mu, \nu) = j$. We have $\#\mathcal{L}(j) < j$. For every $j \in \{1, \dots, 3l\}$ (if $\mathcal{L}(j) \neq \emptyset$), let $\mu_1^{(j)}, \dots, \mu_{k(l,j)}^{(j)}$ be the different numbers $\max\{\mu, \nu\}$ with $(\mu, \nu) \in \mathcal{L}(j)$ arranged in the increasing order. We define the functions

$$\mathcal{G}_j^{(l)}(h; \xi) = \begin{cases} (h^{\mu_1^{(j)}} - \xi)^{-\frac{1}{2}} & \text{if } \xi < h^{\mu_1^{(j)}}, \\ (h^{\mu_k^{(j)}} - \xi)^{-\frac{1}{2}} & \text{if } h^{\mu_{k-1}^{(j)}} < \xi < h^{\mu_k^{(j)}}, \quad k = 2, \dots, k(l, j), \\ 0 & \text{if } \xi > h^{\mu_{k(l,j)}^{(j)}}. \end{cases}$$

For all $\xi \in \mathbb{R} \setminus \{h, h^2, \dots, h^l\}$, we have $\mathcal{G}_j^{(l)}(h; \xi) \leq \mathcal{G}^{(l)}(h; \xi)$. Hence, for all $\Delta > 0$,

$$\max_{\tau \in \mathbb{R}} \int_{\tau}^{\tau+\Delta} \mathcal{G}_j^{(l)}(h; \xi) d\xi \leq T_{l,h}(\Delta) < 5\sqrt{\Delta}. \quad (5.31)$$

Using (5.1), (5.30), and (5.31), we obtain the following estimates (the prime above the summation sign means that we omit the summands with $Y_\delta(\widehat{V}; a_j) = 0$):

$$\sum_{j=1}^{3l'} a_j Y_\delta^{-1}(\widehat{V}; a_j) \frac{1}{(\Xi - 1)\varkappa_1} \times \quad (5.32)$$

$$\begin{aligned}
& \times \int_{\mathfrak{K}_1}^{\Xi \mathfrak{K}_1} \left(\max_{(\mu, \nu) \in \mathcal{L}(j)} \sum_{\substack{n \in \Lambda^* : \\ \pi |n_\perp| \leq \mathfrak{K} + h \max\{\mu, \nu\}}} \frac{\|(\widehat{V}_{[a_j]})_n\|^2}{\sqrt{\mathfrak{K} + 2h \max\{\mu, \nu\} - \pi |n_\perp|}} \right) d\mathfrak{K} \leq \\
& \leq \sum_{j=1}^{3l}' a_j Y_\delta^{-1}(\widehat{V}; a_j) \frac{1}{(\Xi - 1)\mathfrak{K}_1} \int_{\mathfrak{K}_1}^{\Xi \mathfrak{K}_1} \left(\sum_{n \in \Lambda^*} \mathcal{G}_j^{(l)}(h; \pi |n_\perp| - \mathfrak{K}) \|(\widehat{V}_{[a_j]})_n\|^2 \right) d\mathfrak{K} \leq \\
& \leq \frac{5}{\sqrt{(\Xi - 1)\mathfrak{K}_1}} \sum_{j=1}^{3l}' a_j Y_\delta^{-1}(\widehat{V}; a_j) \sum_{n \in \Lambda^*} \|(\widehat{V}_{[a_j]})_n\|^2 \leq \\
& \leq \frac{5M}{\sqrt{(\Xi - 1)\mathfrak{K}_1}} v^{-1}(K) \sum_{j=1}^{3l}' a_j Y_\delta^{-1}(\widehat{V}; a_j) \|(\widehat{V}_{[a_j]})_n\|_{L^2(K; \mathcal{M}_M)}^2 \leq \\
& \leq \frac{5M}{\sqrt{(\Xi - 1)\mathfrak{K}_1}} v^{-1}(K) \sum_{j=1}^{+\infty} (\ln a_j)^{-1-\delta}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \sum_{j=1}^{3l} 2^{-j} \sum_{(\mu, \nu) \in \mathcal{L}(j)} \frac{1}{(\Xi - 1)\mathfrak{K}_1} \times \\
& \times \int_{\mathfrak{K}_1}^{\Xi \mathfrak{K}_1} \left(\sum_{\substack{n \in \Lambda^* : \\ \frac{1}{2}\mathfrak{K}_1 < \pi |n_\perp| \leq \mathfrak{K} + h \max\{\mu, \nu\}}} \frac{\|\widehat{V}_n\|^2}{\sqrt{\mathfrak{K} + 2h \max\{\mu, \nu\} - \pi |n_\perp|}} \right) d\mathfrak{K} \leq \\
& \leq \sum_{j=1}^{3l} 2^{-j} \sum_{(\mu, \nu) \in \mathcal{L}(j)} \frac{1}{(\Xi - 1)\mathfrak{K}_1} \times \\
& \times \sum_{\substack{n \in \Lambda^* : \\ \mathfrak{K}_1 < 2\pi |n_\perp|}} \left(\int_{[\mathfrak{K}_1, \Xi \mathfrak{K}_1] \cap [\pi |n_\perp| - h \max\{\mu, \nu\}, +\infty)} \frac{d\mathfrak{K}}{\sqrt{\mathfrak{K} + 2h \max\{\mu, \nu\} - \pi |n_\perp|}} \right) \|\widehat{V}_n\|^2 \leq \\
& \leq \left(\sum_{j=1}^{3l} j 2^{-j} \right) \frac{1}{(\Xi - 1)\mathfrak{K}_1} \left(\int_{\mathfrak{K}_1}^{\Xi \mathfrak{K}_1} \frac{d\mathfrak{K}}{\sqrt{\mathfrak{K} - \mathfrak{K}_1}} \right) \sum_{\substack{n \in \Lambda^* : \\ \mathfrak{K}_1 < 2\pi |n_\perp|}} \|\widehat{V}_n\|^2 \leq \\
& \leq \frac{4}{\sqrt{(\Xi - 1)\mathfrak{K}_1}} \sum_{\substack{n \in \Lambda^* : \\ \mathfrak{K}_1 < 2\pi |n_\perp|}} \|\widehat{V}_n\|^2.
\end{aligned} \tag{5.33}$$

Let us denote

$$c_5 = c_5(M, \Lambda, \Xi, h; \widehat{V}, \{r_j\}) \doteq \frac{2}{\sqrt{\Xi - 1}} \left(2 + \frac{5}{2} M v^{-1}(K) \sum_{j=1}^{+\infty} (\ln a_j)^{-1-\delta} \right).$$

Then (5.32) and (5.33) imply that there is a number $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$ such that for all $\mu, \nu = 1, \dots, l$,

$$\sum_{\substack{n \in \Lambda^* : \\ \pi|n_\perp| \leq \varkappa + h^{\max\{\mu, \nu\}}}} \frac{\|(\widehat{V}_{[a_j]})_n\|^2}{\sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}} \leq c_5 \varkappa_1^{-\frac{1}{2}} a_j^{-1} Y_\delta(\widehat{V}; a_j), \quad (5.34)$$

$$\sum_{\substack{n \in \Lambda^* : \\ \frac{1}{2} \varkappa_1 < \pi|n_\perp| \leq \varkappa + h^{\max\{\mu, \nu\}}}} \frac{\|\widehat{V}_n\|^2}{\sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}} \leq 2^j c_5 \varkappa_1^{-\frac{1}{2}} \sum_{\substack{n \in \Lambda^* : \\ \varkappa_1 < 2\pi|n_\perp|}} \|\widehat{V}_n\|^2, \quad (5.35)$$

where $a_j = h^{\frac{1}{3}j} r_j$, $j = j(\mu, \nu)$. For every number $j \in \{1, \dots, 3l\}$ (if the number \varkappa_1 is fixed), from (5.28) and (5.35) it follows that for all ordered pairs $(\mu, \nu) \in \mathcal{L}(j)$, all vector functions $\psi \in L^2(K; \mathbb{C}^M)$, and for the number $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$ chosen as above, the estimate

$$\begin{aligned} & \|\widehat{P}_\mu^{(+)} \widehat{V} \widehat{P}_\nu^{(-)} \psi\| \leq \\ & \leq (2c_4 v(K))^{\frac{1}{2}} h^{\frac{1}{2}j} \left((3\Xi \sqrt{\varkappa_1} \sum_{\substack{n \in \Lambda^* : \\ \frac{1}{2} \varkappa_1 < \pi|n_\perp| \leq \varkappa + h^{\max\{\mu, \nu\}}}} \frac{\|\widehat{V}_n\|^2}{\sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}})^{\frac{1}{2}} + \right. \\ & \quad \left. + (\sqrt{\varkappa_1} \sum_{\substack{n \in \Lambda^* : \\ 2\pi|n_\perp| \leq \varkappa_1}} \frac{2\pi|n_\perp| \|\widehat{V}_n\|^2}{\varkappa_1 \sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}})^{\frac{1}{2}} \right) \|\widehat{P}_\nu^{(-)} \psi\| \leq \\ & \leq (2c_4 v(K))^{\frac{1}{2}} h^{\frac{1}{2}j} \left((3 \cdot 2^j \Xi c_5 \sum_{\substack{n \in \Lambda^* : \\ \varkappa_1 < 2\pi|n_\perp|}} \|\widehat{V}_n\|^2)^{\frac{1}{2}} + (\sqrt{2} \sum_{\substack{n \in \Lambda^* : \\ 2\pi|n_\perp| \leq \varkappa_1}} \frac{2\pi|n_\perp|}{\varkappa_1} \|\widehat{V}_n\|^2)^{\frac{1}{2}} \right) \|\widehat{P}_\nu^{(-)} \psi\| \end{aligned}$$

holds. Let $\varepsilon'' \doteq \frac{1}{3} \varepsilon h^{-1} \min\{1, \pi|\gamma|^{-1}\}$. We choose a number $j_0 = j_0(\varepsilon'') \in \mathbb{N}$ (also dependent on $v(K)$, c_5 , Ξ , \widehat{V} , h , $\{r_j\}$) such that $r_j \leq \frac{1}{2} \varepsilon''$ and

$$(6\Xi c_4 c_5 v(K))^{\frac{1}{2}} (r_j^{-1} Y_\delta(\widehat{V}; a_j))^{\frac{1}{2}} \leq \frac{1}{2} \varepsilon''$$

for all $j > j_0$. Since

$$\sum_{\substack{n \in \Lambda^* : \\ \varkappa_1 < 2\pi|n_\perp|}} \|\widehat{V}_n\|^2 \rightarrow 0, \quad \sum_{\substack{n \in \Lambda^* : \\ 2\pi|n_\perp| \leq \varkappa_1}} \frac{2\pi|n_\perp|}{\varkappa_1} \|\widehat{V}_n\|^2 \rightarrow 0 \quad \text{as } \varkappa_1 \rightarrow +\infty,$$

from (5.36) it follows that there is a number $\widetilde{\varkappa}_0 = \widetilde{\varkappa}_0(M, \Lambda, \Xi, h, \widehat{V}, \{r_j\}; j_0, \varepsilon'') > 2h$ such that for all $\varkappa_1 \geq \widetilde{\varkappa}_0$, for the numbers $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$ chosen as above, for all numbers $\mu, \nu = 1, \dots, l$ with $j(\mu, \nu) \leq j_0$ (where $l = l(h, \varkappa_1) \in \mathbb{N}$), and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$ we have

$$\|\widehat{P}_\mu^{(+)} \widehat{V} \widehat{P}_\nu^{(-)} \psi\| \leq \varepsilon'' h^{\frac{1}{3}j(\mu, \nu)} \|\widehat{P}_\nu^{(-)} \psi\|. \quad (5.37)$$

At the same time, if $\varkappa_1 \geq \tilde{\varkappa}_0 > 2h$ and the number $[\varkappa_1, \Xi \varkappa_1] \ni \varkappa$ is chosen as above, then for all numbers $\mu, \nu = 1, \dots, l$ with $j = j(\mu, \nu) > j_0$, and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$, taking into account estimates (5.29), (5.34), and the definition of the number j_0 , we derive

$$\begin{aligned} & \|\widehat{P}_\mu^{(+)} \widehat{V} \widehat{P}_\nu^{(-)} \psi\| \leq \|\widehat{P}_\mu^{(+)} (\widehat{V} - \widehat{V}_{[a_j]}) \widehat{P}_\nu^{(-)} \psi\| + \|\widehat{P}_\mu^{(+)} \widehat{V}_{[a_j]} \widehat{P}_\nu^{(-)} \psi\| \leq \\ & \leq \left(a_j + h^{\frac{1}{2}j} \left(6 \Xi c_4 \sqrt{\varkappa_1} v(K) \sum_{\substack{n \in \Lambda^* : \\ \pi|n_\perp| \leq \varkappa + h^{\max\{\mu, \nu\}}}} \frac{\|(\widehat{V}_{[a_j]})_n\|^2}{\sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}} \right)^{\frac{1}{2}} \right) \|\widehat{P}_\nu^{(-)} \psi\| \leq \\ & \leq h^{\frac{1}{3}j} \left(r_j + (6 \Xi c_4 c_5 v(K))^{\frac{1}{2}} (r_j^{-1} Y_\delta(\widehat{V}; a_j))^{\frac{1}{2}} \right) \|\widehat{P}_\nu^{(-)} \psi\| \leq \varepsilon'' h^{\frac{1}{3}j} \|\widehat{P}_\nu^{(-)} \psi\|, \end{aligned}$$

that is, estimate (5.37) is valid for all $\mu, \nu = 1, \dots, l$.

From (5.3), (5.4), and (5.37) (for all $\mu, \nu = 1, \dots, l$ and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$) we deduce the following estimate:

$$\begin{aligned} & \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\mu^{(+)} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi\| \leq \varepsilon'' (h \max\{1, \frac{|\gamma|}{\pi}\}) h^{\frac{1}{3}j(\mu, \nu)} h^{-\frac{\mu+\nu}{2}} \|\widehat{P}_\nu^{(-)} \psi\| = \\ & = \frac{\varepsilon}{3} h^{\frac{1}{3} \min\{\mu, \nu\}} h^{-\frac{\mu+\nu}{6}} \|\widehat{P}_\nu^{(-)} \psi\| = \frac{\varepsilon}{3} h^{-\frac{1}{6}|\mu-\nu|} \|\widehat{P}_\nu^{(-)} \psi\|. \end{aligned}$$

Whence

$$\begin{aligned} & \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \psi\|^2 = \sum_{\mu=1}^l \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\mu^{(+)} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \left(\sum_{\nu=1}^l \widehat{P}_\nu^{(-)} \right) \psi\|^2 \leq \\ & \leq \sum_{\mu=1}^l \left(\sum_{\nu=1}^l \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\mu^{(+)} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi\| \right)^2 \leq \left(\frac{\varepsilon}{3} \right)^2 \sum_{\mu=1}^l \left(\sum_{\nu=1}^l h^{-\frac{1}{6}|\mu-\nu|} \|\widehat{P}_\nu^{(-)} \psi\| \right)^2 \leq \\ & \leq \left(\frac{\varepsilon}{3} \right)^2 \sum_{\mu=1}^l \left(\sum_{\nu=1}^l h^{-\frac{1}{6}|\mu-\nu|} \right) \left(\sum_{\nu=1}^l h^{-\frac{1}{6}|\mu-\nu|} \|\widehat{P}_\nu^{(-)} \psi\|^2 \right) \leq \\ & \leq \left(\frac{1 + h^{-\frac{1}{6}}}{1 - h^{-\frac{1}{6}}} \right) \left(\frac{\varepsilon}{3} \right)^2 \sum_{\nu=1}^l \left(\sum_{\mu=1}^l h^{-\frac{1}{6}|\mu-\nu|} \right) \|\widehat{P}_\nu^{(-)} \psi\|^2 \leq \\ & \leq \left(\frac{1 + h^{-\frac{1}{6}}}{1 - h^{-\frac{1}{6}}} \right)^2 \left(\frac{\varepsilon}{3} \right)^2 \sum_{\nu=1}^l \|\widehat{P}_\nu^{(-)} \psi\|^2 \leq (\varepsilon)^2 \|\widehat{P}^{(-)} \psi\|^2. \end{aligned}$$

To complete the proof, it remains to put $\psi = \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \varphi$, $\varphi \in L^2(K; \mathbb{C}^M)$. Theorem 5.1 is proved. \square

Now, let us use estimates (5.14) – (5.17), (5.20) – (5.22), and (5.26), (5.27) (conditions (5.11) and (5.23) are fulfilled because

$$\|\widehat{V}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} = \|\widehat{V}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty)} = 0).$$

We choose the number $\varkappa_0 \doteq \max \{ \varkappa'_0(\varepsilon), \varkappa'_0((4\Xi h + 1)^{-\frac{1}{2}} \varepsilon), \varkappa''_0, \tilde{\varkappa}_0 \} > 2h$. Then, for all $\varkappa_1 \geq \varkappa_0$, for the number $\varkappa \in [\varkappa_1, \Xi \varkappa_1]$ chosen in Theorem 5.1, for all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in \tilde{H}^1(K; \mathbb{C}^M)$ we get

$$\begin{aligned}
& \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ \widehat{V} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^- \widehat{V} \varphi\|^2 \leq \\
& \leq 3 \left(\|\widehat{G}_-^{-\frac{1}{2}} P^{(+)} \widehat{V} P^{(+)} \varphi\|^2 + \|\widehat{G}_-^{-\frac{1}{2}} P^{(+)} \widehat{V} P^{(-)} \varphi\|^2 + \|\widehat{G}_-^{-\frac{1}{2}} P^{(+)} \widehat{V} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|^2 \right) + \\
& + 3 \left(\|\widehat{G}_+^{-\frac{1}{2}} P^{(-)} \widehat{V} P^{(+)} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} P^{(-)} \widehat{V} P^{(-)} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} P^{(-)} \widehat{V} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|^2 \right) + \\
& 3 \left(\|\widehat{G}_-^{-\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} P^{(+)} \varphi\|^2 + \|\widehat{G}_-^{-\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} P^{(-)} \varphi\|^2 + \|\widehat{G}_-^{-\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|^2 \right) \leq \\
& \leq 9(\varepsilon)^2 \left(\|\widehat{G}_+^{\frac{1}{2}} P^{(+)} \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} P^{(-)} \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|^2 \right) \leq \\
& \leq (\varepsilon')^2 \left(\|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2 \right)
\end{aligned} \tag{5.38}$$

(where $l = l(h, \varkappa_1)$). This completes the proof of Theorem 1.5.

6 Proof of Theorem 1.6

In what follows, we use the assumptions and the notation from Section 5 in the case where $\Xi = 1$ (and $\varkappa_1 = \varkappa$). We assume that the vectors $k \in \mathbb{R}^3$ satisfy the condition $|(k, \gamma)| = \pi$. The number h is chosen (and fixed) in Section 5 (in the end of the proof of Theorem 1.6, we shall put $h = 64$). Fix a number $\varepsilon' > 0$ and put $\varepsilon = \frac{1}{2\sqrt{6}} \varepsilon'$. Choose a number $\tilde{c}_1 = \tilde{c}_1(h) > 0$ such that $\tilde{c}_1 \leq \frac{1}{8\sqrt{6}} c_3$ and $\tilde{c}_1 < \frac{1}{8\sqrt{6}} C^{-1} (2h^2 + 1)^{-1} < C^{-1}$, where $c_3 = c_3(h)$ is the constant from Lemma 5.3 and $C = C(3)$ is the constant from inequality (0.6). If $\widehat{V}^{(s)} \in L_w^3(K; \mathcal{S}_M^{(s)})$ and

$$\|\widehat{V}^{(s)}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \leq \tilde{c}_1 \varepsilon', \quad s = 0, 1,$$

then inequalities (5.11) and (5.23) (for $\Xi = h$) hold for the function $\widehat{V} = \widehat{V}^{(0)} + \widehat{V}^{(1)}$.

Assume that (for $\Xi = 1$) $\varkappa = \varkappa_1 \geq 2h^2$ (then $l = l(h, \varkappa) \geq 2$ and $2h^{l-1} < h^l \leq \frac{1}{2} \varkappa$). We define the functions

$$\Theta(h, \varkappa; t) = \begin{cases} 1 & \text{if } t \leq h^{l-1}, \\ 2 - h^{-l+1}t & \text{if } h^{l-1} < t \leq 2h^{l-1}, \\ 0 & \text{if } t > 2h^{l-1}, \end{cases}$$

and the operators $\widehat{\Theta} = \widehat{\Theta}(h, k; \varkappa)$ that take vector functions $\psi \in L^2(K; \mathbb{C}^M)$ to the vector functions

$$\widehat{\Theta} \psi = \sum_{N \in \Lambda^*} \Theta(h, \varkappa; G_N^-(k; \varkappa)) \psi_N e^{2\pi i(N, x)}.$$

The following theorem is a key point in the proof of Theorem 1.6.

Theorem 6.1. *Let $d = 3$, $\gamma \in \Lambda \setminus \{0\}$, $\sigma \in (0, 2]$. Then for any Λ -periodic matrix function*

$$\widehat{V}^{(s)} = \sum_{q=1}^{Q_s} \widehat{V}_q^{(s)}, \quad s = 0, 1,$$

with $\widehat{V}_q^{(s)} \in L_w^3(K; \mathcal{S}_M^{(s)})$, $\beta_{\gamma, \sigma}(0; \widehat{V}_q^{(s)}) < +\infty$, $q = 1, \dots, Q_s$, for which the essential supports $\text{supp } \widehat{V}_q^{(s)}$ do not intersect for different q (for $s = 0$ and $s = 1$, separately), and for any $\delta > 0$ there is a number $\tilde{\kappa}'_0(\delta) = \tilde{\kappa}'_0(M, \Lambda, |\gamma|, h, \sigma, \widehat{V}^{(0)}, \widehat{V}^{(1)}; \delta) \geq 2h^2$ such that for all $\kappa \geq \tilde{\kappa}'_0(\delta)$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\varphi \in L^2(K; \mathbb{C}^M)$ the inequalities

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta} \widehat{V}^{(s)} \widehat{P}^- \widehat{\Theta} \varphi\| \leq c_6 (\delta + \max_{q=1, \dots, Q_s} \beta_{\gamma, \sigma}(\widehat{V}_q^{(s)})) \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{(-)} \varphi\|, \quad s = 0, 1,$$

hold, where $c_6 = c_6(h, \sigma) > 0$ (see (6.14)).

Theorem 6.1 is proved in the end of this Section.

Proof of Theorem 1.6. First let us obtain estimates which are similar to estimates (5.14) – (5.17), (5.20) – (5.22), and (5.26) used in the proof of Theorem 1.5. We assume that $\kappa \geq \kappa_0''' \geq 2h^2$. A few additional lower bounds on the number κ_0''' will be given below. For the vectors $k \in \mathbb{R}^3$ we suppose that $|(k, \gamma)| = \pi$, and for the number $l = l(h, \kappa) \in \mathbb{N} \setminus \{1\}$ we have $2h^l \leq \kappa < 2h^{l+1}$ (the number h satisfies the conditions from Section 5: $h \geq 64$ and $h > 2\pi d(K^*)$). Let $\kappa_0''' \geq \kappa_0'(\varepsilon)$. By Lemma 5.3, estimate (5.12) holds. Hence, from (5.14), (5.15), and (5.20) it follows that for all $\varphi \in L^2(K; \mathbb{C}^M)$ the following estimates are fulfilled:

$$\|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}^{(-)} \widehat{V} \widehat{P}^{\mp} \widehat{\Theta} \varphi\| \leq \varepsilon \|\widehat{G}_{\mp}^{\frac{1}{2}} \widehat{P}^{(\mp)} \varphi\|, \quad (6.1)$$

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{V} \widehat{P}^+ \widehat{\Theta} \varphi\| \leq \varepsilon \|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^{(+)} \varphi\|. \quad (6.2)$$

From (5.5) (setting $\Xi = h$ and replacing l by $l - 1$), for all $N \in \Lambda^* \setminus \mathcal{K}(h^{l-1})$ we obtain

$$G_N^-(k; \kappa) > (4h^2 + 1)^{-1} G_N^+(k; \kappa).$$

Therefore, from Lemma 5.3, for $\kappa_0''' \geq \kappa'((4h^2 + 1)^{-\frac{1}{2}} \varepsilon)$, (by analogy with estimates (5.16) and (5.17)) we get

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \widehat{V} \widehat{P}^{\mp} \widehat{\Theta} \varphi\| \leq \varepsilon \|\widehat{G}_{\mp}^{\frac{1}{2}} \widehat{P}^{(\mp)} \varphi\|, \quad \varphi \in L^2(K; \mathbb{C}^M), \quad (6.3)$$

and from (5.19), where ε is replaced by $(4h^2 + 1)^{-\frac{1}{2}} \varepsilon$, (by analogy with estimates (5.16) and (5.17)) we deduce the inequalities

$$\|\widehat{G}_{\mp}^{-\frac{1}{2}} \widehat{P}^{(\pm)} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \varphi\| \leq \varepsilon \|\widehat{G}_{\mp}^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \varphi\|, \quad \varphi \in \widetilde{H}^1(K; \mathbb{C}^M). \quad (6.4)$$

Now, let $\kappa_0''' \geq \kappa_0''$, where the number κ_0'' is chosen for $\Xi = h$. Changing l to $l - 1$ in estimate (5.24), we have

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \widehat{V} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \varphi\| \leq \varepsilon \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \varphi\|, \quad \varphi \in \widetilde{H}^1(K; \mathbb{C}^M). \quad (6.5)$$

Under the conditions of Theorem 6.1, we put $\delta = \frac{1}{4}\varepsilon c_6^{-1}$ and assume that the number \varkappa_0''' satisfies the last lower estimate: $\varkappa_0''' \geq \tilde{\varkappa}_0'(\delta)$. Choose a constant

$$\tilde{c}_1' = \tilde{c}_1'(h, \sigma) = \delta(\varepsilon')^{-1} = \frac{1}{8\sqrt{6}} c_6^{-1}.$$

From Theorem 6.1 (for $\varkappa \geq \varkappa_0'''$) it follows that

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta} \widehat{V} \widehat{P}^- \widehat{\Theta} \varphi\| \leq \varepsilon \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^{(-)} \varphi\|, \quad \varphi \in L^2(K; \mathbb{C}^M). \quad (6.6)$$

Finally, from (6.1) – (6.6), for all $\varkappa \geq \varkappa_0'''$, all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ (by analogy with (5.38)) we obtain estimate (1.15):

$$\begin{aligned} & \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_*^+ \widehat{V} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} \widehat{P}_*^- \widehat{V} \varphi\|^2 \leq \\ & \leq 3 (\|\widehat{G}_-^{-\frac{1}{2}} P^{(+)} \widehat{V} P^+ \widehat{\Theta} \varphi\|^2 + 2 \|\widehat{G}_-^{-\frac{1}{2}} P^+ \widehat{\Theta} \widehat{V} P^- \widehat{\Theta} \varphi\|^2 + \|\widehat{G}_-^{-\frac{1}{2}} P^{(+)} \widehat{V} (\widehat{I} - \widehat{\Theta}) \varphi\|^2) + \\ & + 3 (\|\widehat{G}_+^{-\frac{1}{2}} P^{(-)} \widehat{V} P^+ \widehat{\Theta} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} P^{(-)} \widehat{V} P^- \widehat{\Theta} \varphi\|^2 + \|\widehat{G}_+^{-\frac{1}{2}} P^{(-)} \widehat{V} (\widehat{I} - \widehat{\Theta}) \varphi\|^2) + \\ & 3 (\|\widehat{G}_-^{-\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} P^+ \widehat{\Theta} \varphi\|^2 + 2 \|\widehat{G}_-^{-\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \widehat{V} P^- \widehat{\Theta} \varphi\|^2 + \|\widehat{G}_-^{-\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \widehat{V} (\widehat{I} - \widehat{\Theta}) \varphi\|^2) \leq \\ & \leq 3(\varepsilon)^2 (3 \|\widehat{G}_+^{\frac{1}{2}} P^{(+)} \varphi\|^2 + 5 \|\widehat{G}_-^{\frac{1}{2}} P^{(-)} \varphi\|^2 + 3 \|\widehat{G}_-^{\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^{l-1})} \varphi\|^2) \leq \\ & \leq 3(\varepsilon)^2 (6 \|\widehat{G}_+^{\frac{1}{2}} P^{(+)} \varphi\|^2 + 8 \|\widehat{G}_-^{\frac{1}{2}} P^{(-)} \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|^2) \leq \\ & \leq 24(\varepsilon)^2 (\|\widehat{G}_+^{\frac{1}{2}} P^{(+)} \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} P^{(-)} \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} P^{\Lambda^* \setminus \mathcal{K}(h^l)} \varphi\|^2) \leq \\ & \leq (\varepsilon')^2 (\|\widehat{G}_+^{\frac{1}{2}} \widehat{P}^+ \varphi\|^2 + \|\widehat{G}_-^{\frac{1}{2}} \widehat{P}^- \varphi\|^2). \end{aligned}$$

It remains to remove the technical lower bound $h > 2\pi d(K^*)$ for the number h . Under the linear transformations $x \rightarrow \lambda x$, $x \in \mathbb{R}^3$, where $\lambda > 0$, we have $\gamma \rightarrow \lambda \gamma$ (the vector $\tilde{e} \in S_1(\gamma)$ does not change), $K^* \rightarrow \lambda^{-1} K^*$, $d(K^*) \rightarrow \lambda^{-1} d(K^*)$, and

$$\varepsilon' \rightarrow \lambda \varepsilon', \quad \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})} \rightarrow \lambda \|\widehat{W}\|_{L_w^3(K; \mathcal{M}_M)}^{(\infty, \text{loc})}, \quad \beta_{\gamma, \sigma}(\widehat{W}) \rightarrow \lambda \beta_{\gamma, \sigma}(\widehat{W}).$$

Therefore conditions 3 and 4 from Theorem 1.6, and estimate (1.15) do not change under such transformations. Choosing the number $\lambda > 0$ such that $h = 64 > \lambda^{-1} \cdot 2\pi d(K^*)$, we conclude that we can take the universal constant $\tilde{c}_1 = \tilde{c}_1(64)$ and the constant $\tilde{c}_1' = \tilde{c}_1'(64, \sigma)$ dependent only on σ . Theorem 1.6 is proved.

Theorem 6.2. *Let $d = 3$, $\gamma \in \Lambda \setminus \{0\}$, $\sigma \in (0, 2]$. Suppose that $\widehat{W} \in L^2(K; \mathcal{M}_M)$, $\beta_{\gamma, \sigma}(0; \widehat{W}) < +\infty$, and for a.e. $x \in \mathbb{R}^3$ the matrices $\widehat{W}(x)$ commute with all orthogonal projections $\widehat{P}_{\tilde{e}}^\pm$, $\tilde{e} \in S_1(\gamma)$ (in particular, we may consider the functions $\widehat{W} = \widehat{W}^{(0)} + \widehat{W}^{(1)}$, $\widehat{W}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$). Then for any $\delta > 0$ there is a number $\varkappa^\sharp > 2h$ such that for all $\varkappa \geq \varkappa^\sharp$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$ the inequality*

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{W} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \psi\| \leq \frac{1}{2} c_6 (\delta + \beta_{\gamma, \sigma}(\widehat{W})) \|\psi\|, \quad (6.7)$$

holds, where $c_6 = c_6(h, \sigma) > 0$ is the constant from Theorem 6.1 (see (6.14)).

Proof. To start with, we assume that $\sigma \in (0, \frac{1}{4}]$. Let $\varkappa > 2h$ and let $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$. We shall derive upper bounds for the norms

$$\| \widehat{P}_\mu^{(+)} \widehat{W} \widehat{P}_\nu^{(-)} \psi \|, \quad \mu, \nu = 1, \dots, l, \quad \psi \in L^2(K; \mathbb{C}^M).$$

Since the Fourier coefficients \widehat{W}_N , $N \in \Lambda^*$, commute with all orthogonal projections $\widehat{P}_\varepsilon^\pm$, $\varepsilon \in S_1(\gamma)$, we can use estimate (5.28). For $\mu, \nu \in \{1, \dots, l\}$ and $R \in [2\pi d(K^*), 2\varkappa]$, the function \widehat{W} can be represented in the form

$$\widehat{W}(x) = \widehat{W}_{\mu, \nu}^{[0]}(R; x) + \widehat{W}_{\mu, \nu}^{[1]}(R; x), \quad x \in \mathbb{R}^3,$$

where

$$\widehat{W}_{\mu, \nu}^{[0]}(R; x) = \sum_{\substack{N \in \Lambda^* : \\ 2\pi|N_\perp| \leq R}} \widehat{W}_N e^{2\pi i(N, x)}, \quad \widehat{W}_{\mu, \nu}^{[1]}(R; x) = \sum_{\substack{N \in \Lambda^* : \\ R < 2\pi|N_\perp|}} \widehat{W}_N e^{2\pi i(N, x)}.$$

By (5.28), we get

$$\begin{aligned} & \| \widehat{P}_\mu^{(+)} \widehat{W}_{\mu, \nu}^{[0]}(R; \cdot) \widehat{P}_\nu^{(-)} \psi \|^2 \leq \\ & \leq v(K) \varkappa^{-2} \sum_{N \in \mathcal{K}_\mu} \left(\sum_{\substack{n \in \Lambda^* : \\ 2\pi|n_\perp| \leq R, \\ N-n \in \mathcal{K}_\nu}} 2\pi|n_\perp| \|\widehat{W}_n\| \|(\widehat{P}^- \psi)_{N-n}\| \right)^2 \leq \\ & \leq v(K) \varkappa^{-2} \left(\sum_{\substack{n \in \Lambda^* : \\ 2\pi|n_\perp| \leq R, \\ \pi|n_\parallel| \leq h^{\max\{\mu, \nu\}}} 2\pi|n_\perp| \|\widehat{W}_n\| \right)^2 \left(\sum_{N \in \Lambda^*} \|(\widehat{P}_\nu^{(-)} \psi)_N\|^2 \right) \leq \\ & \leq R^2 \varkappa^{-2} \left(\sum_{\substack{n \in \Lambda^* : \\ 2\pi|n_\perp| \leq R, \\ \pi|n_\parallel| \leq h^{\max\{\mu, \nu\}}} 1 \right) \left(\sum_{n \in \Lambda^*} \|\widehat{W}_n\|^2 \right) \|\widehat{P}_\nu^{(-)} \psi\|^2 \leq \\ & \leq 3\pi^{-2} M R^4 h^{\max\{\mu, \nu\}} \varkappa^{-2} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)}^2 \|\widehat{P}_\nu^{(-)} \psi\|^2. \end{aligned} \tag{6.8}$$

The following estimate is also a consequence of (5.28):

$$\begin{aligned} & \| \widehat{P}_\mu^{(+)} \widehat{W}_{\mu, \nu}^{[1]}(R; \cdot) \widehat{P}_\nu^{(-)} \psi \|^2 \leq \\ & \leq c_4 v^{-1}(K) h^{\mu+\nu+\min\{\mu, \nu\}} \varkappa^{-\frac{1}{2}} \beta_{\gamma, \sigma}^2(R; \widehat{W}) \times \\ & \times \left(\sum_{\substack{n \in \Lambda^* : \\ R < 2\pi|n_\perp| \leq 2\varkappa+2h^{\max\{\mu, \nu\}}, \\ \pi|n_\parallel| \leq h^{\max\{\mu, \nu\}}} \frac{(2\pi|n_\perp|)^{-2(1-\sigma)} (2\pi|n|)^{-2\sigma}}{(\pi|n_\perp| + h^{\max\{\mu, \nu\}}) \sqrt{\varkappa + 2h^{\max\{\mu, \nu\}} - \pi|n_\perp|}} \right) \|\widehat{P}_\nu^{(-)} \psi\|^2. \end{aligned} \tag{6.9}$$

Under the condition $R < 2h^{\max\{\mu, \nu\}}$, we have

$$\sum_{\substack{n \in \Lambda^* : \\ R < 2\pi|n_\perp| \leq 2h^{\max\{\mu, \nu\}}, \\ \pi|n_\parallel| \leq h^{\max\{\mu, \nu\}}} (2\pi|n_\perp|)^{-2(1-\sigma)} (2\pi|n|)^{-2\sigma} \leq$$

$$\begin{aligned}
&\leq \frac{4}{(2\pi)^3 v(K^*)} \left(4\pi (h^{\max\{\mu,\nu\}} + \pi d(K^*)) + \right. \\
&\quad \left. + \int_R^{2h^{\max\{\mu,\nu\}} + 2\pi d(K^*)} \frac{2\pi\xi d\xi}{\xi^{2(1-\sigma)}} \int_{-2h^{\max\{\mu,\nu\}} - 2\pi d(K^*)}^{2h^{\max\{\mu,\nu\}} + 2\pi d(K^*)} \frac{d\eta}{(\xi^2 + \eta^2)^\sigma} \right) \leq \\
&\leq \pi^{-2} v^{-1}(K^*) \left(3h^{\max\{\mu,\nu\}} + \int_R^{3h^{\max\{\mu,\nu\}}} \frac{d\xi}{\xi^{1-2\sigma}} \int_{-3h^{\max\{\mu,\nu\}}}^{3h^{\max\{\mu,\nu\}}} \frac{d\eta}{(\xi^2 + \eta^2)^\sigma} \right) < \\
&< 8\pi^{-2} \sigma^{-1} v^{-1}(K^*) h^{\max\{\mu,\nu\}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{n \in \Lambda^* : \\ h^{\max\{\mu,\nu\}} < \pi|n_\perp| \leq \varkappa + h^{\max\{\mu,\nu\}}, \\ \pi|n_\parallel| \leq h^{\max\{\mu,\nu\}}}} \frac{(2\pi|n_\perp|)^{2\sigma} (2\pi|n|)^{-2\sigma}}{(2\pi|n_\perp|)^3 \sqrt{\varkappa + 2h^{\max\{\mu,\nu\}} - \pi|n_\perp|}} \leq \\
&\leq \sum_{\substack{n \in \Lambda^* : \\ h^{\max\{\mu,\nu\}} < \pi|n_\perp| \leq \varkappa + h^{\max\{\mu,\nu\}}, \\ \pi|n_\parallel| \leq h^{\max\{\mu,\nu\}}}} \frac{1}{(2\pi|n_\perp|)^3} \frac{1}{\sqrt{\varkappa + 2h^{\max\{\mu,\nu\}} - \pi|n_\perp|}} \leq \\
&\leq \frac{4}{(2\pi)^3 v(K^*)} \left(\frac{3}{2} \right)^{\frac{5}{2}} (h^{\max\{\mu,\nu\}} + \pi d(K^*)) \times \\
&\quad \times \int_{2h^{\max\{\mu,\nu\}} - 2\pi d(K^*)}^{2\varkappa + 2h^{\max\{\mu,\nu\}} + 2\pi d(K^*)} \frac{2\pi\xi d\xi}{\xi^3 \sqrt{\varkappa + 2h^{\max\{\mu,\nu\}} - \frac{1}{2}\xi}} \leq \\
&\leq \frac{\sqrt{3}}{\pi^2} \left(\frac{3}{2} \right)^4 v^{-1}(K^*) h^{\max\{\mu,\nu\}} \times \\
&\quad \times \left(\varkappa^{-\frac{1}{2}} \left| \begin{array}{c} \xi = \varkappa \\ \xi = h^{\max\{\mu,\nu\}} \end{array} \right. - \frac{1}{\xi} + 2\varkappa^{-2} \left| \begin{array}{c} \xi = 2\varkappa + 3h^{\max\{\mu,\nu\}} \\ \xi = \varkappa \end{array} \right. - \sqrt{2\varkappa + 4h^{\max\{\mu,\nu\}} - \xi} \right) < \\
&< 34\pi^{-2} v^{-1}(K^*) \varkappa^{-\frac{1}{2}}.
\end{aligned}$$

Hence, (6.9) yields

$$\begin{aligned}
&\| \widehat{P}_\mu^{(+)} \widehat{W}_{\mu,\nu}^{[1]}(R; \cdot) \widehat{P}_\nu^{(-)} \psi \|^2 \leq \\
&\leq 25\pi^{-2} c_4 \sigma^{-1} h^{\mu+\nu+\min\{\mu,\nu\}} \varkappa^{-1} \beta_{\gamma,\sigma}^2(R; \widehat{W}) \|\widehat{P}_\nu^{(-)} \psi\|^2.
\end{aligned} \tag{6.10}$$

From (6.8) and (6.10) (also see (5.3) and (5.4)), for all $\varkappa > 2h$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, all vector functions $\psi \in L^2(K; \mathbb{C}^M)$, and all numbers $R \in [2\pi d(K^*), 2\varkappa]$ we obtain

$$\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{W} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \psi \| \leq \sum_{\mu, \nu=1}^l \| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\mu^{(+)} \widehat{W} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi \| \leq \tag{6.11}$$

$$\begin{aligned}
&\leq \sqrt{\frac{|\gamma|}{\pi}} \sum_{\nu=1}^l \|\widehat{P}_1^{(+)} (\widehat{W}_{1,\nu}^{[0]}(R; \cdot) + \widehat{W}_{1,\nu}^{[1]}(R; \cdot)) \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi\| + \\
&+ \sum_{\mu=2}^l \sum_{\nu=1}^l h^{-\frac{\mu-1}{2}} \|\widehat{P}_\mu^{(+)} (\widehat{W}_{\mu,\nu}^{[0]}(R; \cdot) + \widehat{W}_{\mu,\nu}^{[1]}(R; \cdot)) \widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi\| \leq \\
&\leq \sqrt{\frac{|\gamma|}{\pi}} \sum_{\nu=1}^l \left(\frac{\sqrt{3M}}{\pi} R^2 h^{\frac{1}{2}\nu} \varkappa^{-1} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} + \right. \\
&\quad \left. + \frac{5}{\pi\sqrt{\sigma}} c_4^{\frac{1}{2}} h^{1+\frac{1}{2}\nu} \varkappa^{-\frac{1}{2}} \beta_{\gamma,\sigma}(R; \widehat{W}) \right) \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi\| + \\
&+ \sum_{\mu=2}^l \sum_{\nu=1}^l h^{-\frac{\mu-1}{2}} \left(\frac{\sqrt{3M}}{\pi} R^2 h^{\frac{1}{2} \max\{\mu,\nu\}} \varkappa^{-1} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} + \right. \\
&\quad \left. + \frac{5}{\pi\sqrt{\sigma}} c_4^{\frac{1}{2}} h^{\frac{1}{2}(\mu+\nu+\min\{\mu,\nu\})} \varkappa^{-\frac{1}{2}} \beta_{\gamma,\sigma}(R; \widehat{W}) \right) \|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}_\nu^{(-)} \psi\| \leq \\
&\leq \frac{\sqrt{h}}{\pi} \left(\frac{|\gamma|}{\pi} + 2(l-1) \sqrt{\frac{|\gamma|}{\pi}} \right) (\sqrt{3M} R^2 \varkappa^{-1} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} + \\
&\quad + 5 c_4^{\frac{1}{2}} \sigma^{-\frac{1}{2}} h \varkappa^{-\frac{1}{2}} \beta_{\gamma,\sigma}(R; \widehat{W})) \|\widehat{P}_\nu^{(-)} \psi\| + \\
&\quad + \frac{h}{\pi} \sum_{\mu,\nu=2}^l (\sqrt{3M} R^2 h^{-\frac{1}{2} \min\{\mu,\nu\}} \varkappa^{-1} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} + \\
&\quad + 5 c_4^{\frac{1}{2}} \sigma^{-\frac{1}{2}} h^{\frac{1}{2} \min\{\mu,\nu\}} \varkappa^{-\frac{1}{2}} \beta_{\gamma,\sigma}(R; \widehat{W})) \|\widehat{P}_\nu^{(-)} \psi\|.
\end{aligned}$$

On the other hand, $l \leq (\ln^{-1} h) \ln \frac{\varkappa}{2}$ and

$$\frac{h}{\varkappa} \sum_{\mu,\nu=1}^l h^{-\frac{1}{2} \min\{\mu,\nu\}} \leq h^{\frac{1}{2}} l^2 \varkappa^{-1} \leq (h^{\frac{1}{2}} \ln^{-1} h) \varkappa^{-1} \ln \frac{\varkappa}{2}, \quad (6.12)$$

$$\begin{aligned}
&\frac{h}{\sqrt{\varkappa}} \sum_{\mu,\nu=2}^l h^{\frac{1}{2} \min\{\mu,\nu\}} = \frac{h}{\sqrt{\varkappa}} h^{\frac{1}{2}l} \sum_{\mu_1,\nu_1=0}^{l-2} h^{-\frac{1}{2} \max\{\mu_1,\nu_1\}} \leq \\
&\leq \frac{h}{\sqrt{2}} \sum_{\mu_1=0}^{+\infty} (2\mu_1+1) h^{-\frac{1}{2}\mu_1} = \frac{h}{\sqrt{2}} (1+h^{-\frac{1}{2}})(1-h^{-\frac{1}{2}})^{-2} < 2h
\end{aligned}$$

(because $h > 16$). Therefore (for $\sigma \in (0, \frac{1}{4}]$),

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{W} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \psi\| \leq \quad (6.13)$$

$$\leqslant \left(10 \pi^{-1} c_4^{\frac{1}{2}} \sigma^{-\frac{1}{2}} h \beta_{\gamma, \sigma}(R; \widehat{W}) + \mathfrak{F}(\sigma, h, R, \widehat{W}; \varkappa) \right) \|\widehat{P}^{(-)} \psi\|,$$

where

$$\begin{aligned} \mathfrak{F}(\sigma, h, R, \widehat{W}; \varkappa) &= \pi^{-1} \sqrt{h} (\ln^{-1} h) \left(\ln \frac{\varkappa}{2} \right) \left(1 + \sqrt{\frac{|\gamma|}{\pi}} \right)^2 \times \\ &\times \left(\sqrt{3M} R^2 \varkappa^{-1} \|\widehat{W}\|_{L^2(K; \mathcal{M}_M)} + 5 c_4^{\frac{1}{2}} \sigma^{-\frac{1}{2}} h \varkappa^{-\frac{1}{2}} \beta_{\gamma, \sigma}(R; \widehat{W}) \right). \end{aligned}$$

Now let us suppose that $\sigma \in (0, 2]$. Denote

$$c_6 = 40 \sqrt{2} \pi^{-1} c_4^{\frac{1}{2}} h \sigma^{-\frac{1}{2}} \quad (6.14)$$

and $\sigma' \doteq \min\{\frac{1}{4}, \sigma\}$. Since $(\sigma')^{-\frac{1}{2}} \leqslant 2\sqrt{2} \sigma^{-\frac{1}{2}}$ and $\beta_{\gamma, \sigma'}(R; \widehat{W}) \leqslant \beta_{\gamma, \sigma}(R; \widehat{W})$ (for all $R \geqslant 0$), inequality (6.13) (in which we replace σ by σ') implies the estimate

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(+)} \widehat{W} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^{(-)} \psi\| \leqslant \left(\frac{1}{2} c_6 \beta_{\gamma, \sigma}(R; \widehat{W}) + \mathfrak{F}(\sigma, h, R, \widehat{W}; \varkappa) \right) \|\widehat{P}^{(-)} \psi\|. \quad (6.15)$$

Finally, choose (and fix) a number $R \geqslant 2\pi d(K^*)$ such that

$$\beta_{\gamma, \sigma}(R; \widehat{W}) \leqslant \frac{\delta}{2} + \beta_{\gamma, \sigma}(\widehat{W}).$$

Then, from (6.15) it follows that there is a number $\varkappa^\# = \varkappa^\#(M, \Lambda, |\gamma|, h, \sigma; \widehat{W}, \delta) > 2h$, for which $\varkappa^\# \geqslant \frac{1}{2} R$, such that for all $\varkappa \geqslant \varkappa^\#$, all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\psi \in L^2(K; \mathbb{C}^M)$ the inequality (6.7) holds. This completes the proof of Theorem 6.2.

Let $l' \in \mathbb{N} \setminus \{1\}$ and let $\varkappa \geqslant 2h^{l'+1}$. Then $l = l(h, \varkappa) \geqslant l' + 1 \geqslant 3$. Denote

$$l_1 \doteq l - l' \in \{1, \dots, l - 2\}$$

and define the functions

$$\Theta_{l'}(h, \varkappa; t) = \begin{cases} 1 & \text{if } h^{l_1+1} < t \leqslant h^{l-1}, \\ 2 - h^{-l_1+1} t & \text{if } h^{l-1} < t \leqslant 2h^{l-1}, \\ -1 + 2h^{-l_1-1} t & \text{if } \frac{1}{2} h^{l_1+1} < t \leqslant h^{l_1+1}, \\ 0 & \text{if } t \leqslant \frac{1}{2} h^{l_1+1} \text{ or } t > 2h^{l-1}, \end{cases}$$

and the operators $\widehat{\Theta}_{l'} = \widehat{\Theta}_{l'}(h, k; \varkappa)$ that take a vector function $\psi \in L^2(K; \mathbb{C}^M)$ to the vector function

$$\widehat{\Theta}_{l'} \psi = \sum_{N \in \Lambda^*} \Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) \psi_N e^{2\pi i(N, x)}.$$

Theorem 6.3. *Let $d = 3$, $\gamma \in \Lambda \setminus \{0\}$, $\sigma \in (0, 2]$. Suppose that $\widehat{V} = \widehat{V}^{(0)} + \widehat{V}^{(1)}$, where $\widehat{V}^{(s)} \in L^2(K; \mathcal{S}_M^{(s)})$, $s = 0, 1$, and $\beta_{\gamma, \sigma}(0; \widehat{V}) < +\infty$. Then for any $\varepsilon > 0$ there are numbers $l' = l'(\Lambda, |\gamma|, \sigma, \widehat{V}; \varepsilon) \in \mathbb{N} \setminus \{1\}$ and $\varkappa_0^\sim = \varkappa_0^\sim(M, \Lambda, |\gamma|, h, \sigma, \widehat{V}; \varepsilon) \geqslant 2h^{l'+1}$ such that for all $\varkappa \geqslant \varkappa_0^\sim$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$ the inequality*

$$\|\widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^- \widehat{\Theta} \psi - \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta}_{l'} \widehat{V} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^- \widehat{\Theta}_{l'} \psi\| \leqslant \varepsilon \|\widehat{P}^- \widehat{\Theta} \psi\| \quad (6.16)$$

holds.

Proof. Taking into account the inclusions

$$\widehat{\Theta}_{l'}\varphi \in \mathcal{H}\left(\bigcup_{\mu=l_1+1}^l \mathcal{K}_l\right), \quad (\widehat{\Theta} - \widehat{\Theta}_{l'})\varphi \in \mathcal{H}\left(\bigcup_{\mu=1}^{l_1+1} \mathcal{K}_l\right), \quad \varphi \in L^2(K; \mathbb{C}^M),$$

and estimates (5.3), (5.4), from (6.8) and (6.10), where we put $R = 2\pi d(K^*)$, it follows that for all $\varkappa \geq 2h^{l'+1}$, all vectors $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all vector functions $\psi \in L^2(K; \mathbb{C}^M)$ the following inequalities are valid (see (6.11)):

$$\begin{aligned} & \|\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^+\widehat{\Theta}\widehat{V}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^-\widehat{\Theta}\psi - \widehat{G}_-^{-\frac{1}{2}}\widehat{P}^+\widehat{\Theta}_{l'}\widehat{V}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^-\widehat{\Theta}_{l'}\psi\| \leq \quad (6.17) \\ & \leq \|\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^+(\widehat{\Theta} - \widehat{\Theta}_{l'})\widehat{V}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^-\widehat{\Theta}\psi\| + \|\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^+\widehat{\Theta}_{l'}\widehat{V}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}^-(\widehat{\Theta} - \widehat{\Theta}_{l'})\psi\| \leq \\ & \leq \sum_{\mu=1}^{l_1+1} \sum_{\nu=1}^l \|\widehat{G}_-^{-\frac{1}{2}}\widehat{P}_\mu^{(+)}\widehat{V}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}_\nu^{(-)}\widehat{\Theta}\psi\| + \\ & + \sum_{\mu=l_1+1}^l \sum_{\nu=1}^{l_1+1} \|\widehat{G}_-^{-\frac{1}{2}}\widehat{P}_\mu^{(+)}\widehat{V}\widehat{G}_-^{-\frac{1}{2}}\widehat{P}_\nu^{(-)}(\widehat{\Theta} - \widehat{\Theta}_{l'})\psi\| \leq \\ & \leq \frac{h}{\pi} \max\left\{1, \frac{|\gamma|}{\pi}\right\} \sqrt{3M} (2\pi d(K^*))^2 \|\widehat{V}\|_{L^2(K; \mathcal{M}_M)} \varkappa^{-1} \times \\ & \times \left(\left(\sum_{\mu=1}^{l_1+1} \sum_{\nu=1}^l + \sum_{\mu=l_1+1}^l \sum_{\nu=1}^{l_1+1} \right) h^{-\frac{1}{2} \min\{\mu, \nu\}} \right) \|\widehat{P}^-\widehat{\Theta}\psi\| + \\ & + \frac{h}{\pi} \max\left\{1, \frac{|\gamma|}{\pi}\right\} 5c_4^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \beta_{\gamma, \sigma}(0; \widehat{V}) \varkappa^{-\frac{1}{2}} \times \\ & \times \left(\left(\sum_{\mu=1}^{l_1+1} \sum_{\nu=1}^l + \sum_{\mu=l_1+1}^l \sum_{\nu=1}^{l_1+1} \right) h^{\frac{1}{2} \min\{\mu, \nu\}} \right) \|\widehat{P}^-\widehat{\Theta}\psi\|. \end{aligned}$$

At the same time, the estimates (6.12) and

$$\begin{aligned} & \frac{h}{\sqrt{\varkappa}} \left(\sum_{\mu=1}^{l_1+1} \sum_{\nu=1}^l + \sum_{\mu=l_1+1}^l \sum_{\nu=1}^{l_1+1} \right) h^{\frac{1}{2} \min\{\mu, \nu\}} < \frac{2h}{\sqrt{\varkappa}} \sum_{\mu=1}^{l_1+1} \sum_{\nu=1}^l h^{\frac{1}{2} \min\{\mu, \nu\}} < \\ & < 2h (h^{\frac{1}{2}l} \varkappa^{-\frac{1}{2}}) \sum_{\mu_1=0}^{+\infty} (l' + 2\mu_1) h^{-\frac{1}{2}(l'+\mu_1-1)} < 2\sqrt{2} (l' + 2) h^{\frac{1}{2}(3-l')} \end{aligned}$$

hold (because $h > 4$). Therefore, inequality (6.17) implies that there are numbers $l' = l'(\Lambda, |\gamma|, \sigma, \widehat{V}; \varepsilon) \in \mathbb{N} \setminus \{1\}$ and $\varkappa_0^\sim = \varkappa_0^\sim(M, \Lambda, |\gamma|, h, \sigma, \widehat{V}; \varepsilon) \geq 2h^{l'+1}$ such that inequality (6.16) is fulfilled for all $\varkappa \geq \varkappa_0^\sim$, all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\psi \in L^2(K; \mathbb{C}^M)$. Theorem 6.3 is proved.

Proof of Theorem 6.1. From Theorem 6.3 (under the change $\psi = \widehat{G}_-^{\frac{1}{2}} \widehat{P}^{(-)} \varphi$, $\varphi \in L^2(K; \mathbb{C}^M)$), we see that it suffices to prove that for any numbers $\delta > 0$ and $l' \in \mathbb{N} \setminus \{1\}$ there is a number $\tilde{\varkappa}'_0(\delta, l') = \tilde{\varkappa}'_0(M, \Lambda, |\gamma|, h, \sigma, \widehat{V}^{(0)}, \widehat{V}^{(1)}; \delta, l') \geq 2h^{l'+1}$ such that for all $\varkappa \geq \tilde{\varkappa}'_0(\delta, l')$, all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\psi \in L^2(K; \mathbb{C}^M)$ the inequalities

$$\begin{aligned} & \| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta}_{l'} \widehat{V}^{(s)} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^- \widehat{\Theta}_{l'} \psi \| \leq \\ & \leq c_6 \left(\delta + \max_{q=1, \dots, Q_s} \beta_{\gamma, \sigma}(\widehat{V}_q^{(s)}) \right) \| \widehat{P}^{(-)} \psi \|, \quad s = 0, 1, \end{aligned} \quad (6.18)$$

hold. Fix numbers $\delta > 0$ and $l' \in \mathbb{N} \setminus \{1\}$. For $s = 0, 1$ and $q = 1, \dots, Q_s$, suppose $\mathcal{F}_q^{(s)} \in C^\infty(\mathbb{R}^3; \mathbb{R})$ are Λ -periodic functions such that $0 \leq \mathcal{F}_q^{(s)}(x) \leq 1$ for all $x \in \mathbb{R}^3$, $\mathcal{F}_q^{(s)}(x) = 1$ for $x \in \text{supp } \mathcal{F}_q^{(s)}$, and $\text{supp } \mathcal{F}_{q_1}^{(s)} \cap \text{supp } \mathcal{F}_{q_2}^{(s)} = \emptyset$ for $q_1 \neq q_2$, $s = 0, 1$. Let us denote $\tilde{\delta} = 16h^{-l'-1}$ (then $\tilde{\delta}\varkappa < 32h^{l_1}$, where $l_1 = l - l'$). We define the functions

$$\mathcal{F}_q^{(s,1)}(x) = \sum_{\substack{N \in \Lambda^* : \\ 2\pi|N| \leq \tilde{\delta}\varkappa}} (\mathcal{F}_q^{(s)})_N e^{2\pi i(N, x)}, \quad \mathcal{F}_q^{(s,2)}(x) = \sum_{\substack{N \in \Lambda^* : \\ 2\pi|N| > \tilde{\delta}\varkappa}} (\mathcal{F}_q^{(s)})_N e^{2\pi i(N, x)}, \quad x \in \mathbb{R}^3.$$

For all $\beta \geq 0$ (and all $q = 1, \dots, Q_s$, $s = 0, 1$),

$$\varkappa^\beta \| \mathcal{F}_q^{(s,2)} \|_{L^\infty(\mathbb{R}^3; \mathbb{R})} \rightarrow 0 \quad (6.19)$$

as $\varkappa \rightarrow +\infty$.

In what follows, we shall use the brief notation

$$\widehat{\mathcal{A}}_{l'}^\pm \doteq \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^\pm \widehat{\Theta}_{l'}. \quad (6.20)$$

Lemma 6.1. *There are constants $c_7(h, l'; \mathcal{F}_q^{(s)}) > 0$ such that for all $\varkappa \geq 2h^{l'+1}$, all $k \in \mathbb{R}^3$, and all $\varphi \in L^2(K; \mathbb{C}^M)$ we have*

$$\| (\widehat{\mathcal{A}}_{l'}^\pm \mathcal{F}_q^{(s,1)} - \mathcal{F}_q^{(s,1)} \widehat{\mathcal{A}}_{l'}^\pm) \varphi \| \leq c_7(h, l'; \mathcal{F}_q^{(s)}) \varkappa^{-\frac{3}{2}} \| \varphi \|, \quad (6.21)$$

$q = 1, \dots, Q_s$, $s = 0, 1$.

Proof. The choice of the number $\tilde{\delta}$ implies that

$$(\mathcal{F}_q^{(s,1)} \widehat{\mathcal{A}}_{l'}^\pm \varphi)_N = (\widehat{\mathcal{A}}_{l'}^\pm \mathcal{F}_q^{(s,1)} \varphi)_N = 0$$

for $N \in (\Lambda^* \setminus \mathcal{K}(h^{l_1})) \cup \mathcal{K}(h^{l_1})$,

$$\begin{aligned} & (\widehat{\mathcal{A}}_{l'}^\pm \mathcal{F}_q^{(s,1)} \varphi)_N = \\ & = \sum_{\substack{n \in \Lambda^* : \\ 2\pi|n| \leq \tilde{\delta}\varkappa, \\ N-n \in \mathcal{K}(h^{l_1}) \setminus \mathcal{K}(h^{l_1})}} (G_N^-(k; \varkappa))^{-\frac{1}{2}} \widehat{P}_{\widehat{e}(k+2\pi N)}^\pm \Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) (\mathcal{F}_q^{(s)})_n \varphi_{N-n}, \end{aligned} \quad (6.22)$$

$$(\mathcal{F}_q^{(s,1)} \widehat{\mathcal{A}}_{l'}^\pm \varphi)_N = \quad (6.23)$$

$$= \sum_{\substack{n \in \Lambda^* : \\ 2\pi|n| \leq \delta \varkappa, \\ N-n \in \mathcal{K}(h^l) \setminus \mathcal{K}(h^{l_1})}} (G_{N-n}^-(k; \varkappa))^{-\frac{1}{2}} \widehat{P}_{\tilde{e}(k+2\pi(N-n))}^\pm \Theta_{l'}(h, \varkappa; G_{N-n}^-(k; \varkappa)) (\mathcal{F}_q^{(s)})_n \varphi_{N-n}$$

for $N \in \mathcal{K}(h^l) \setminus \mathcal{K}(h^{l_1})$. Furthermore,

$$\begin{aligned} & (G_N^-(k; \varkappa))^{-\frac{1}{2}} \widehat{P}_{\tilde{e}(k+2\pi N)}^\pm \Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) - \\ & - (G_{N-n}^-(k; \varkappa))^{-\frac{1}{2}} \widehat{P}_{\tilde{e}(k+2\pi(N-n))}^\pm \Theta_{l'}(h, \varkappa; G_{N-n}^-(k; \varkappa)) = \\ & = ((G_N^-(k; \varkappa))^{-\frac{1}{2}} - (G_{N-n}^-(k; \varkappa))^{-\frac{1}{2}}) \widehat{P}_{\tilde{e}(k+2\pi N)}^\pm \Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) + \\ & + (G_{N-n}^-(k; \varkappa))^{-\frac{1}{2}} (\widehat{P}_{\tilde{e}(k+2\pi N)}^\pm - \widehat{P}_{\tilde{e}(k+2\pi(N-n))}^\pm) \Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) + \\ & + (G_{N-n}^-(k; \varkappa))^{-\frac{1}{2}} \widehat{P}_{\tilde{e}(k+2\pi(N-n))}^\pm (\Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) - \Theta_{l'}(h, \varkappa; G_{N-n}^-(k; \varkappa))). \end{aligned} \quad (6.24)$$

For $N, N-n \in \mathcal{K}(h^l) \setminus \mathcal{K}(h^{l_1})$, we have

$$|(G_N^-(k; \varkappa))^{-\frac{1}{2}} - (G_{N-n}^-(k; \varkappa))^{-\frac{1}{2}}| \leq \frac{1}{2} h^{-\frac{3}{2}l_1} \cdot 2\pi|n|,$$

$$\|\widehat{P}_{\tilde{e}(k+2\pi N)}^\pm - \widehat{P}_{\tilde{e}(k+2\pi(N-n))}^\pm\| \leq \frac{2\pi|n|}{\varkappa},$$

$$|\Theta_{l'}(h, \varkappa; G_N^-(k; \varkappa)) - \Theta_{l'}(h, \varkappa; G_{N-n}^-(k; \varkappa))| \leq 2h^{-l_1-1} \cdot 2\pi|n|$$

(see (1.4), (5.2), and the definition of the functions $\Theta_{l'}(h, \varkappa; \cdot)$). Therefore (6.22) – (6.24) yield

$$\|((\widehat{\mathcal{A}}_{l'}^\pm \mathcal{F}_q^{(s,1)} - \mathcal{F}_q^{(s,1)} \widehat{\mathcal{A}}_{l'}^\pm) \varphi)_N\| \leq 3\sqrt{2} h^{\frac{3}{2}(l'+1)} \varkappa^{-\frac{3}{2}} \sum_{\substack{n \in \Lambda^* : \\ 2\pi|n| \leq \delta \varkappa}} 2\pi|n| \cdot |(\mathcal{F}_q^{(s)})_n| \cdot \|\varphi_{N-n}\|.$$

From this we obtain that estimates (6.21) hold with constants

$$c_7(h, l'; \mathcal{F}_q^{(s)}) = 3\sqrt{2} h^{\frac{3}{2}(l'+1)} \sum_{n \in \Lambda^*} 2\pi|n| \cdot |(\mathcal{F}_q^{(s)})_n|.$$

Lemma 6.1 is proved. \square

If $\widehat{W} \in L_w^3(K; \mathcal{M}_M)$, then (see (0.6) and (1.1)) there is a constant $c_8 = c_8(\Lambda, |\gamma|; \widehat{W}) > 0$ such that for all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\varphi \in \widetilde{H}^1(K; \mathbb{C}^M)$ the following inequality is satisfied:

$$\|\widehat{W}\varphi\| \leq c_8 \left\| \sum_{j=1}^3 \widehat{\alpha}_j \left(k_j - i \frac{\partial}{\partial x_j} \right) \varphi \right\|.$$

Therefore, for all vector functions $\varphi \in \mathcal{H}(\mathcal{K}(h^l))$,

$$\|\widehat{W}\varphi\| \leq \frac{3}{2} c_8 \varkappa \|\varphi\|. \quad (6.25)$$

Now let us obtain inequality (6.18) (for sufficiently large numbers $\varkappa \geq \tilde{\varkappa}'_0(\delta, l') \geq 2h^{l'+1}$). From (6.19), (6.25), and Lemma 6.1 it follows that there exists a number $\tilde{\varkappa}'_0(\delta, l') \geq 2h^{l'+1}$ (dependent also on Λ , $|\gamma|$, h , σ , on the numbers Q_s , and on the functions $\widehat{V}_q^{(s)}$, $\mathcal{F}_q^{(s)}$) such that for all $\varkappa \geq \tilde{\varkappa}'_0(\delta, l')$, all $k \in \mathbb{R}^3$ with $|(k, \gamma)| = \pi$, and all $\psi \in L^2(K; \mathbb{C}^M)$ we have

$$\begin{aligned}
\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta}_{l'} \widehat{V}^{(s)} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^- \widehat{\Theta}_{l'} \psi \|^2 &= \| \widehat{\mathcal{A}}_{l'}^+ \sum_{q=1}^{Q_s} (\mathcal{F}_q^{(s,1)} + \mathcal{F}_q^{(s,2)}) \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 \leq \\
&\leq \frac{8}{7} \| \widehat{\mathcal{A}}_{l'}^+ \sum_{q=1}^{Q_s} \mathcal{F}_q^{(s,1)} \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 + \frac{1}{14} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq \frac{9}{7} \| \sum_{q=1}^{Q_s} \mathcal{F}_q^{(s,1)} \widehat{\mathcal{A}}_{l'}^+ \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 + \frac{2}{14} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq \frac{10}{7} \| \sum_{q=1}^{Q_s} \mathcal{F}_q^{(s)} \widehat{\mathcal{A}}_{l'}^+ \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 + \frac{3}{14} \delta^2 c_6^2 \|\psi\|^2 = \\
&= \frac{10}{7} \sum_{q=1}^{Q_s} \| \mathcal{F}_q^{(s)} \widehat{\mathcal{A}}_{l'}^+ \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 + \frac{3}{14} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq \frac{11}{7} \sum_{q=1}^{Q_s} \| \mathcal{F}_q^{(s,1)} \widehat{\mathcal{A}}_{l'}^+ \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 + \frac{4}{14} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq \frac{12}{7} \sum_{q=1}^{Q_s} \| \widehat{\mathcal{A}}_{l'}^+ \mathcal{F}_q^{(s,1)} \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \psi \|^2 + \frac{5}{14} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq \frac{13}{7} \sum_{q=1}^{Q_s} \| \widehat{\mathcal{A}}_{l'}^+ \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \mathcal{F}_q^{(s,1)} \psi \|^2 + \frac{6}{14} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq 2 \sum_{q=1}^{Q_s} \| \widehat{\mathcal{A}}_{l'}^+ \widehat{V}^{(s)} \widehat{\mathcal{A}}_{l'}^- \mathcal{F}_q^{(s)} \psi \|^2 + \frac{1}{2} \delta^2 c_6^2 \|\psi\|^2
\end{aligned}$$

(we use the notation (6.20)). Finally, (perhaps picking a larger number $\tilde{\varkappa}'_0(\delta, l')$ which now may also depend on M) these estimates and Theorem 6.2 imply that inequalities (6.18) hold:

$$\begin{aligned}
&\| \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^+ \widehat{\Theta}_{l'} \widehat{V}^{(s)} \widehat{G}_-^{-\frac{1}{2}} \widehat{P}^- \widehat{\Theta}_{l'} \psi \|^2 \leq \\
&\leq \frac{1}{2} c_6^2 \sum_{q=1}^{Q_s} (\delta + \beta_{\gamma, \sigma}(\widehat{V}_q^{(s)}))^2 \| \mathcal{F}_q^{(s)} \psi \|^2 + \frac{1}{2} \delta^2 c_6^2 \|\psi\|^2 \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} c_6^2 (\delta + \max_{q=1,\dots,Q_s} \beta_{\gamma,\sigma}(\widehat{V}_q^{(s)}))^2 \sum_{q=1}^{Q_s} \|\mathcal{F}_q^{(s)} \psi\|^2 + \frac{1}{2} \delta^2 c_6^2 \|\psi\|^2 \leq \\
&\leq c_6^2 (\delta + \max_{q=1,\dots,Q_s} \beta_{\gamma,\sigma}(\widehat{V}_q^{(s)}))^2 \|\psi\|^2, \quad s = 0, 1.
\end{aligned}$$

This completes the proof of Theorem 6.1.

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